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LOGIC OF SHEAVES OF STRUCTURES ON A LOCALE

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Abstract

We study the logic of sheaves of structures over topological spaces developed by Caicedo from the point of view of pointless topology showing that Caicedo's results, in particular the Generic Model Theorem, still hold for sheaves of structures on a locale. We describe, using Fourman and Scott's techniques, the sheafification functor from the category of presheaves to the category of sheaves of structures on a locale, obtaining an isomorphism between the category of complete Ω-structures and the category of sheaves of structures on Ω. In this context, we present some connections between logic and geometry.

Keywords: logic, sheaves, topology without points, locale theory.

CONTENTS

Chapter 1

Introduction

From 1907 on dutch mathematician L.E.J. Brouwer made a strong effort in order to create an alternative to classic mathematics known as intuitionistic mathematics. From an intuitionistic point of view mathematics are mental constructions, in particular, a statement is valid only when it has been given a proof of it. Logic laws such as the excluded middle $(A \vee \neg A)$, or the equivalence between an statement and its double negation $(A \equiv \neg\neg A)$ are no longer true in general under this interpretation, their truthfulness or falsehood depending on the statement A.

Differences between classic and intuitionistic mathematics go beyond logic, in fact Brouwer himself never dealt with logical issues. It was Arend Heyting, his best student, who proposed both propositional and first order calculi for intuitionistic logic in 1930. The first models of this propositional calculus where of the same nature as the Boolean Algebras, that is to say an algebraic copy of the calculus axioms, they are called Heyting Algebras . Topological spaces are more interesting models of this calculus, their lattices of open sets being Heyting Algbras. Locales are complete Heyting algebras, these algebras are designed to resemble the lattice of open sets of a topological space. What makes this approach particular is that its basic concept is not points, but rather neighborhoods of points.

There have been several proposals of first order semantics for the intuitionistic calculus, each of which satisfies a completeness theorem. Among the most interesting we find Kripke models, topoi (Kripke-Joyal semantics) and sheaves of structures on topological spaces as studied by Xavier Caicedo [Ca95]. All of these objects can be seen as variable structures.

Topological sheaves of structures occupy a flexible place in between Kripke models and semantics of topoi. The epistemic principle that prompts Caicedo's work is continuity of truth, i.e, that every statement satisfied in a point of a space should be satisfied in a neighborhood of the point. This postulate finds a deep basis in the fact that every observable phenomenon occurs extended in time and space, hence punctual properties are no more than ideal limits of observable properties. This principle holds in the logic of topological sheaves studied by Caicedo.

All points are no more than ideal limits of its surroundings, hence it is worth to ask what does it happen with the logic of sheaves of structures if we exchange the underlying topological space for a locale, that is to say, if we forget points and focus on neighborhoods. A category similar to the one of sheaves of structures on a locale have been studied by Fourman and Scott [FS79], in their work they develop adequate techniques in order to describe the sheafification functor for presheaves of sets on a locale.

The main objective of this work is to generalize Caicedo's results [Ca95] to the scope of sheaves of structures on a locale. In order to attain this in the first chapter we establish preliminary results about intuicionistic logic and locale theory. In the first section of the second chapter we describe the sheafification functor for preshaves of structures on a locale using the tools developed by Fourman and Scott [FS79], in particular the category of complete Ω-structures which is proved to be isomorphic to the category of sheaves of structures on a locale. In the second section of the second chapter we define semantics for sheaves of structures on a locale. In this context we present natural generalizations of the theorems in [Ca95], including the Generic Model Theorem. The last result is described by Caicedo as the Fundamental Theorem of Model Theory, because it has as consequences Łoś ultraproducts theorem and the completeness theorem for first order logic among others. We present a characterization of connected locales in terms of the semantics defined and we sketch a characterization of compact locales using Fourman and Scott's semantics, both valid for the topological case.

Chapter 2

Preliminaries

2.1 Intuitionistic Logic

Intuitionistic logic arises from the logical study of a good portion of the work of dutch mathematician and philosopher L.E.J. Brouwer. For Brouwer, mathematics are mental constructions in which the language emerge at first as accessory and it is just necessary in order to achieve practical purposes such as memorization and communication. Logic depends then on mathematics, because it studies the regularities in mathematical language, but mathematics is independent from logic. No wonder why Brouwer never took care of the logical aspects of his work[vD13, p. 96].

For Brouwer, mathematical objects are mental constructions, so from the fact that it is contradictory that a property holds for all objects in a certain class does not follow that there is one object in the class for which the property does not hold. This last object should be given by a construction. Brouwer was opposed to Hilbert for whom the existence of a mathematical object meant merely that this existence was non-contradictory, for the dutch mathematician existence meant existence in certain conditions. Following this train of thought, since mathematical truths are also mental constructions (proofs), it is not certain that the excluded middle holds for a statement A, that the assertion $A \vee \neg A$ holds would mean that we have a proof of A or a proof of $\neg A$. This is shown with great clarity in the Brouwerian counter examples to the excluded middle. For $n \in \mathbb{N}$, let $R(n)$ be the assertion: the digits of the decimal expansion of π are all 9 in between the positions $n - 9$ and n. Consider the sequence

$$a_n = \begin{cases} 2^{-n}, & \text{if for all } k \leq n, \neg R(k). \\ (-2)^{-k}, & \text{if } R(k), k \leq n \text{ and for all } p < k, \neg R(p). \end{cases}$$

This sequence converges to a real a, designated by Brouwer as pendular number. But we have no evidence which proves that there is an n which satisfies $R(n)$, nor, in case of its existence, do we have a clue about its being even or odd, then this a does not satisfy $a = 0 \vee a \neq 0$, even the trichotomy principle fails to hold for this a [vD13, p. 284].

Although the concept of existence in Brouwer has aspects in common with the one of Poincar, Borel, Baire and Hadamard, the so called semi-intuitionistics, Brower's rejection of the excluded middle makes a big difference. Another important issue that comes between Brouwer and his predecessors is the acceptance of the existence of sequences of objects that are not governed by a law or a finite definition. This is a key point for the development of his intuitionistic mathematics whose publication started in 1918 with the article: *Founding Mathematics independently of the logical theorem of the excluded middle,* a confusing name if we consider that in all the series of articles the logical principle is not even mentioned, except for the title. Brouwer's alternative mathematics is not just a logical restriction of classical mathematics but a whole another mathematical universe in which all real functions are continuous and the continuum (real line) is not only connected but it can not be decomposed in any two disjoint sets. In particular one has $\neg\forall x(x \in \mathbb{Q} \vee x \notin \mathbb{Q})$, so that the excluded middle is not only not always true but even contradictory. Brower's most famous theorem, the fixed point theorem, needs to be reformulated as it turns out to be false in his original version. For a more detailed account of Brouwer's intuitionistic mathematics we refer the reader to [vD13] sections 8.6 and 10.2-10.4.

As we have said, Brouwer never took care of the logical aspects of his work, it was one of his closest friends, Gerrit Mannoury, who proposed in 1927 the problem of formalizing the logical principles used in intuitionistic mathematics, in the same fashion it had been done with the classical principles thanks to the effort of Frege, Russel, Whitehead, Peano, Hilbert et al. [vD13, p. 500]. The problem, proposed for the Dutch Mathematical Society's annual contest, was solved by Arend Heyting, who will publish his solution in 1930. In this article propositional and first order logic as well as arithmetic and set theory are formalized in an intuitionistic form. It is worth mentioning that years before Glivenko and Kolmogorov had formalized fragments of the propositional and first order intuitionistic calculus. From 1931 on Heyting presented an epistemic interpretation of the logical connectives known as *proof interpretation:* it is argued that the validity of any assertion in mathematical language depends upon a construction or proof, in this way $\exists x\phi(x)$ is true when there is a construction of an object d, such that $\phi(d)$. The implication $\phi \rightarrow \psi$ is true when there is a construction that transforms any proof of ϕ in a proof of ψ, the negation $\neg\phi$ is interpreted as $\phi \rightarrow \bot$ where $\bot$ is contradiction, that has no proof [vD02, p. 6]. A germ of this interpretation was present in Brouwer's doctoral dissertation [vD13, p. 96].

Heyting algebras are algebraic copies of the axioms of intuitionistic propositional calculus, as proposed by Heyting, in the same way Boolean algebras arise from classical logic. Classical logic is obtained from intuitionistic logic by adding the excluded middle or any other equivalent principle as an axiom. We define the intuitionistic propositional calculus, here we use the logical connectives $\neg, \rightarrow, \vee$ y $\wedge$ which are non interdefinable in this setting[Bo94, p. 2].

1.1 Definition. The *intuitionistic propositional calculus* H has axioms [vD02, p. 9]:

H1 $\phi \rightarrow (\psi \rightarrow \phi)$,

H2 $(\phi \rightarrow (\psi \rightarrow \varphi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \varphi))$,

H3 $\phi \rightarrow (\psi \rightarrow (\phi \wedge \psi))$,

H4 $(\phi \wedge \psi) \rightarrow \phi$,

H5 $(\phi \wedge \psi) \rightarrow \psi$,

H6 $\phi \rightarrow (\phi \vee \psi)$,

H7 $\psi \rightarrow (\phi \vee \psi)$,

H8 $(\phi \rightarrow \varphi) \rightarrow ((\psi \rightarrow \varphi) \rightarrow ((\phi \vee \psi) \rightarrow \varphi))$,

H9 $(\phi \rightarrow \psi) \rightarrow ((\phi \rightarrow \neg\psi) \rightarrow \neg\phi)$,

H10 $\neg\phi \rightarrow (\phi \rightarrow \psi)$.

where ϕ, ψ and φ are formulas of the intuitionistic calculus. The only law of deduction is Modus Ponens, we infer $\vdash_H \psi$ from $\vdash_H \phi$ and $\vdash_H \phi \rightarrow \psi$.

We also define the first order intuitionistic calculus.

1.2 Definition. The *first order intuitionistic calculus* H^*, arises from the intuitionistic propositional calculus H by adding the following axioms where ϕ y ψ are first order formulas:

- $\forall x \phi(x) \rightarrow \phi(t)$,
- $\phi(t) \rightarrow \exists x \phi(x)$,

where t is a free term for x in ϕ, besides we add the following rules of deduction:

- If $\vdash_{H^*} \phi \to \psi(x)$, and x does not occur free in ϕ, then $\vdash_{H^*} \phi \to \forall x \psi(x)$.
- If $\vdash_{H^*} \phi(x) \to \psi$, and x does not occur free in ψ, then $\vdash_{H^*} \exists x \phi(x) \to \psi$.

The difference between classical and intuitionistic calculus occurs in the propositional level. The deduction theorem turns out to be important since it allows us to characterize the relation between the connectives $\wedge$ and $\to$. In what follows we use the fact that a preorder can be seen as a category.

1.3 Lemma. [Bo94, p. 5] *If we order propositional statements by making $\phi \leq \psi$ iff $\vdash_H \phi \to \psi$, the functor $\phi \wedge -$ defined by $\varphi \mapsto \phi \wedge \varphi$, admits the functor $\phi \to -$ as a right adjoint. Thus we have a Galois connection, this means that for all propositional statements ϕ, ψ and φ, $\phi \wedge \psi \leq \varphi$ iff $\phi \leq \psi \to \varphi$.*

The last result allows us to give a simple definition of a semantics for intuitionistic propositional calculus through Heyting algebras.

1.4 Definition. [Bo94, p. 5] A *Heyting algebra* $\mathcal{H}$ is a lattice, with different top and bottom, in wich for all $b \in \mathcal{H}$ the functor:

$$- \wedge b : \mathcal{H} \to \mathcal{H}, a \mapsto a \wedge b,$$

has a right adjoint, denoted by

$$b \Rightarrow - : \mathcal{H} \to \mathcal{H}, a \mapsto b \Rightarrow a.$$

We use 0 and 1 to denote the bottom and the top element respectively. We use $\neg a$ to denote $a \Rightarrow 0$.

It is known that for preorders, the adjunctions are Galois connections, in this way the satisfaction of the Galois connection

$$a \wedge b \leq c \text{ iff } a \leq b \Rightarrow c$$

for a, b, c in the lattice $\mathcal{H}$ is equivalent to the adjunction.

Heyting algebras turn out to be a natural semantics for intuitionistic propositional calculus. As an example, for H1, if $a, b \in \mathcal{H}$, then $a \Rightarrow (b \Rightarrow a) = 1$. Relations that equal 1 in all Heyting algebras are exactly the deducible from H. The following properties of Heyting algebras will be used often, so we list them next.

1.5 Proposition. [Bo94, p. 5-8] *Let $\mathcal{H}$ be a Heyting algebra, where a, b y c are elements of $\mathcal{H}$, the following hold:*

- $b \Rightarrow c = \bigvee\{a : a \wedge b \leq c\} = \max\{a : a \wedge b \leq c\}$.
- $\neg b = \bigvee\{a : a \wedge b = 0\} = \max\{a : a \wedge b = 0\}$.
- $\neg(a \vee b) = \neg a \wedge \neg b$
- $a \wedge \neg a = 0$
- $a \leq b$ *iff* $a \Rightarrow b = 1$.
- $a \leq \neg\neg a$.
- *If* $a \leq b$, *then* $\neg b \leq \neg a$.
- $\neg 0 = 1, \neg 1 = 0$.
- $\neg a = \neg\neg\neg a$.

2.2 Locales

The history of intuitionistic logic interweaves at this point with the history of topology without points. The idea behind locales (complete Heyting algebras) is to study topological spaces by studying his lattice of open sets. Although from the definition of topological space by Hausdorff it was evident that points played a secondary role and what was really important to understand the space structure was the behavior of the surroundings of each point and not of all subsets in which the point is included, moving from the study of topology to the study of the underlying lattices of opens needed the development of lattice theory and the important impulse of Marshall Stone's work. He proved that the dual category of Boolean algebras is isomorphic to the category of Hausdorff, compact, totally disconnected topological spaces. The last result suggested that the lattice of a topological space contained more information than was expected, and this motivated the work of mathematicians like Nbeling, Mc Kinsey and Tarski who from different perspectives tried to understand topology without points. Still, it was Isbell, the first to introduce the term locale, who showed that the study of the category of locales could be of great use in finding that products, subspaces and dense spaces behave differently in this case. It was found that this approach enables to give constructive proofs of theorems that in the classic case of topological space require strongly non intuitionistic principles (axiom of choice) as in the case of Tychonoff's theorem. For more details on the history of the development of this branch of topology we refer the reader to [Jo01, p. 835]. Next we define locales, the only example for the moment being the lattice of open sets of a topological space.

2.1 Definition. [Bo94, p. 14] A *locale* $\mathcal{L}$ is a complete lattice in which the following distributive law holds for elements of $\mathcal{L}$:

$$a \wedge \big(\bigvee_{i \in I} b_i\big) = \bigvee_{i \in I} (a \wedge b_i).$$

2.2 Proposition. [Bo94, p. 14] *A lattice $\mathcal{L}$ is a locale iff it is a complete Heyting algebra.*

Proof. If $\mathcal{L}$ is a locale and $a \in \mathcal{L}$, the functor $a \wedge -$ preserves arbitrary colimits (suprema), since $\mathcal{L}$ is a small category, $a \wedge -$ has a right adjoint by the adjoint functor theorem. If $\mathcal{L}$ is a complete Heyting algebra, the existence of a right adjoint for $a \wedge -$ implies that it preserves arbitrary colimits (suprema). □

If a and b are elements of a locale $\mathcal{L}$, its product is just $a \wedge b$. Since $a \wedge -$ has a right adjoint, each locale is a cartesian closed category with exponentiation given by $b^a = a \Rightarrow b$.

2.3 Example. Let $(X, \mathcal{T})$ be a topological space, $\mathcal{T}$ is a locale for the partial order given by inclusion. If U is an open set,$\neg U$ is the interior of $X \setminus U$ and $\neg\neg U$ is the interior of the closure of U. In the locale of open sets of $\mathbb{R}$ we can see some counterexamples to classical logic principles. For example if $U = (-\infty, 0)$ and $V = (0, +\infty)$ we see that $U \vee \neg U = U \vee V \neq \mathbb{R}$, $\neg\neg(U \vee V) = \mathbb{R} \neq U \vee V$.

The study of lattices of topological spaces admits another justification. In [Ca95], Caicedo argues that observable phenomena are presented extended in time and space, or at least in space. Points, just as punctual statements, are ideal limits of space and observable phenomena. Caicedo makes use of the following example (which was studied one hundred years before by American philosopher Charles S. Peirce [Pe, P. 367]): suppose there is a white sheet with a black region, about the points in the boundary between the two regions it can't be said that they are black nor white, that would mean that the color black or white would happen to be extended in a surrounding of the boundary point, which is impossible. Hence it seems that excluded middle fails to hold in boundary points. The color is just observable in surroundings of the point, not in the point itself which constitutes an idealization of space.

From this perspective it is justified continuity of truth, the logic principle which asserts that all logical statements satisfied in a point should be satisfied in a neighborhood of the point, just as Serre proposed in the middle fifties [FAC,GAGA], in the intersection between algebraic and analytic geometry. The logic of sheaves of structures developed in [Ca95] satisfies this principle. This paradigm justifies the study of topological spaces not through his points, idealizations used in the classical study of topology, but through their lattices of open sets which represent the relations between surroundings of points. This brings us back to intuitionistic ideas because the individualization of each of the points of the continuum is a classical operation that finds no analog in the intuitionistic framework of Brouwer. In this mathematics there are individual points given by a construction, bot these barely form a set of measure zero, the rest of the points (random points not given by a construction) are the ones that give linear continuum its true structure. We continue with the definition of the basic concepts of locale theory that will be needed later on. $\mathcal{L}, \mathcal{M}$ y Ω will denote locales.

2.4 Definition. [Bo94, p. 16] A *morphism of locales* $f : \mathcal{L} \to \mathcal{M}$ consists in a pair of morphisms $f^* : \mathcal{M} \to \mathcal{L}$, $f_* : \mathcal{L} \to \mathcal{M}$ such that:

- f^* is left adjoint to f_*,
- f^* preserves finite infima.

2.5 Lemma. [Bo94, p. 17] *There exists a bijection between morphisms of locales $f : \mathcal{L} \to \mathcal{M}$ and maps $f^* : \mathcal{M} \to \mathcal{L}$ which satisfy:*

- $f^*(\vee_{i \in I} a_i) = \vee_{i \in I} f^*(a_i)$*,*
- $f^*(a \wedge b) = f^*(a) \wedge f^*(b)$
- $f^*(1) = 1$ *for an index set I and $a, b, a_i \in \mathcal{M}$. In this bijection, each f^* gives rise in a unique way to its right adjoint f_*.*

2.6 Example.

Every continuous function between topological spaces$f : (X, \mathcal{T}) \to (Y, \mathcal{Y})$ induces a morphism of locales $f : \mathcal{T} \to \mathcal{M}$ with $f^* = f^{-1}$, because f^{-1} maps open sets in open sets and preserves arbitrary unions and finite intersections.

We can define the category of locales $\mathbf{Loc}$ whose objects are locales and whose morphisms are morphisms of locales. If we define composition by making $(f \circ g)^*$ equal to $g^* \circ f^*$ associativity and existence of identities are satisfied.

2.7 Lemma. [Bo94, p. 15] *Let a be an element of $\mathcal{L}$, then*

$$\uparrow a := \{b \in \mathcal{L} \mid a \leq b\}, \ \downarrow a := \{b \in \mathcal{L} \mid b \leq a\}$$

are both locales with the induced partial order.

2.8 Example.

- Let a be an element of $\mathcal{L}$. The functor $i^* : \mathcal{L} \to \downarrow a$ defined for $b \in \mathcal{L}$ such that $i^*(b) = a \wedge b$ defines a morphism of locales, his right adjoint being i_* defined for $c \leq a$ by $i^*(c) = a \Rightarrow c$. Since i^* is surjective, i is a monomorphism of locales and it is the inclusion of $\downarrow a$ into $\mathcal{L}$. In the case where $\mathcal{L}$ is the lattice of open sets of a topological space and $a \in \mathcal{L}$, $\downarrow a$ turns out to be the lattice of open sets of the space a with the subspace topology.

- The functor $j^* : \mathcal{L} \to \uparrow a$ defined for $b \in L$ by $j^*(b) = b \vee a$ induces a morphism of locales since it preserves arbitrary suprema, finite infima and maxima. Its right adjoint is j_* defined for any $c \geq a$, by $j_*(c) = c$. Again, since j^* is surjective, the induced morphism j is a monomorphism and it is the inclusion of $\uparrow a$ in $\mathcal{L}$. In the case where $\mathcal{L}$ is the lattice of open sets of a topological space and $a \in \mathcal{L}$, the locale $\uparrow a$ coincides with the local of open sets of the complement of a with the subspace topology [Bo94, p. 16].

We can then define open and closed sublocales.

2.9 Definition. [Bo94, p. 18]

- An *open sublocale* of $\mathcal{L}$ is a monomorphism of locales $f : \mathcal{M} \to \mathcal{L}$ isomorphic to the monomorphism $i : \downarrow a \to \mathcal{L}$, for some $a \in \mathcal{L}$.

- A *closed sublocale* of $\mathcal{L}$ is a monomorphism of locales $f : \mathcal{M} \to \mathcal{L}$ isomorphic to the monomorphism $i : \uparrow a \to \mathcal{L}$ for some $a \in \mathcal{L}$.

In order to the define the concept of dense sublocale we will need to consider more subobjects than just the open and closed ones. These subjobjects (which turn out to be regular) are associated with certain morphism that we define next.

2.10 Definition. A *nucleus* on a locale $\mathcal{L}$ is a functor $j : \mathcal{L} \to \mathcal{L}$ which satisfies the following conditions:

(N1) $a \leq j(a)$,

(N2) $jj(a) \leq j(a)$,

(N3) $j(a \wedge b) = j(a) \wedge j(b)$;

for $a, b \in \mathcal{L}$. Given a nucleus j, we define

$$\mathcal{L}^j = \{a \in \mathcal{L} \mid a = j(a)\}.$$

$\mathcal{L}^j$ turns out to be a locale with the inherited order where infima coincide with infima in $\mathcal{L}$ and in order to obtain the supremum of a family we apply j to the supremum of the family in $\mathcal{L}$ [Bo94, p. 29].

A useful characterization of nuclei is the following.

2.11 Lemma. [JoII, p. 481] *A mapping $j : \mathcal{L} \to \mathcal{L}$ is a nucleus iff the following relation is satisfied by any $a, b \in \mathcal{L}$,*

$$(a \Rightarrow j(b)) = (j(a) \Rightarrow j(b)).$$

The three conditions in the definition of nucleus are equivalent to ask that j is a monad [Bo95, p. 189] induced by a morphism of locales, that is to say, by its pair of adjoint functors [Bo94, p. 29]. This can be used to show that each nucleus j induces a morphism of locales $\mathcal{L}^j \to \mathcal{L}$ in which $j : \mathcal{L} \to \mathcal{L}^j$ is the left adjoint and the identity $(a \mapsto a)$ the right adjoint. From the reflexivity of $\mathcal{L}^j$ in $\mathcal{L}$ we can derive the locale structure of $\mathcal{L}^j$ mentioned in the definition. Since the mapping $(a \mapsto a)$ is injective we have that the induced morphism is a monomorphism and so $\mathcal{L}^j$ is a subobject of $\mathcal{L}$. The following proposition tells us a bit more about subobjects induced via nuclei.

2.12 Proposition. [Bo94, p. 31] *Let $f : \mathcal{M} \to \mathcal{L}$ be a morphism of locales. The following conditions are equivalent:*

- *f is a regular monomorphism of locales (the equalizer of a pair of morphisms);*
- *f_* is injective;*
- *f^* is surjective;*
- *$f^* \circ f_* = 1_{\mathcal{M}}$;*
- *f is isomorphic to the morphism $\mathcal{L}^j \to \mathcal{L}$ induced by the nucleus $j = f_* \circ f^*$ in $\mathcal{L}$.*

As a corolary of the above we have epi-regular mono factorizations in the category of locales.

2.13 Corollary. [Bo94, p. 32] *Every morphism of locales factors uniquely as an epimorphism followed by a regular monomorphism.*

2.14 Example.

- Let $a \in \mathcal{L}$, and $i : \downarrow a \to \mathcal{L}$. i is a regular monmorphism since i^* is surjective. The corresponding nucleus, as defined in the above example is $j_a : \mathcal{L} \to \mathcal{L}$ such that for $b \in \mathcal{L}$, $j_a(b) = a \Rightarrow (b \wedge a) = a \Rightarrow b$. Where the last equality holds by the laws of intuitionistic logic.

- Let $a \in \mathcal{L}$, the closed sublocale $i : \uparrow a \to \mathcal{L}$ is also a regular monomorphism, the corresponding nucleus is $j^a : \mathcal{L} \to \mathcal{L}$ such that for $b \in \mathcal{L}$, $j^a(b) = a \vee b$.

- Double negation allows us to define a nucleus $\neg\neg : \mathcal{L} \to \mathcal{L}$ such that for $a \in \mathcal{L}$, $\neg\neg(a) = \neg\neg a$. The satisfaction of (N1) to (N3) is a follows if we use the intuitionistic calculus. Besides

$$\mathcal{L}^{\neg\neg} = \{\neg a \mid a \in \mathcal{L}\}.$$

2.15 Theorem. [Bo94, p. 33] *The nuclei on a locale, preordered with the pointwise order constitute a locale. The dual of the locale of nuclei is the poset of regular subobjects of the locale $\mathcal{L}$ with the inclusion preorder.*

Proof. We already have a bijection between regular subobjects and nuclei. The pointwise preorder for nuclei j and k is defined as follows:

$$j \leq k \text{ iff for all } a \in \mathcal{L},\, j(a) \leq k(a),$$

$j \leq k$ is equivalent to $\mathcal{L}^k \subseteq \mathcal{L}^j$ as sets, and this is equivalent to $\mathcal{L}^k \subseteq \mathcal{L}^j$ as subobjects.
If $(j_i)_{i \in I}$, is a family of nuclei and $a \in \mathcal{L}$,

$$\left(\bigwedge_{i \in I} j_i\right)(a) = \bigwedge_{i \in I} j_i(a).$$

□

The locale of nuclei has the identity as its bottom, which corresponds to the maximum regular sublocale $\mathcal{L}$, and as its top $j : \mathcal{L} \to \mathcal{L}$ such that $a \in \mathcal{L}$, $j(a) = 1$ which corresponds to the trivial sublocale $\{1\}$. In the locale of nuclei suprema are hard to define, but those are equivalent to arbitrary intersections of families of regular subojects of the locale. If we consider a family of regular subobjects as given by their sets of fixed points $(\mathcal{L}^j)_{j \in I}$, it is possible to show that the intersection set of the family is again a regular sublocale. This can be done using the following characterization of $\mathcal{L}^j$. We will need the concept of exponential ideal.

2.16 Definition. [JoI, p. 62] A subset S of a locale $\mathcal{L}$ is an *exponential ideal* iff for all $b \in S$ and $a \in \mathcal{L}$, we have that $a \Rightarrow b \in S$.

2.17 Proposition. [JoII, p. 481] *Let $\mathcal{L}$ be a locale, there is a bijection between the set of nuclei of $\mathcal{L}$ and the set of exponential ideals of $\mathcal{L}$ which are closed under arbitrary infima.*

Proof. Let j be a nucleus, then $\mathcal{L}^j$ is closed under arbitrary infima being a reflective subcategory of $\mathcal{L}$, in this case, $j : \mathcal{L} \to \mathcal{L}^j$ is right adjoint to the identity because of (N1) and (N2). For the other implication we use a result that we assume without proof [JoI, p. 185] and which tells us that the satisfaction of (N3) by j, that is to say, preservation of finite limits, is equivalent to the fact that $\mathcal{L}^j$ be an exponential ideal. Given an exponential ideal closed under abitrary infima S, we have that the inclusion $S \to \mathcal{L}$ preserves arbitrary limits, and so it has a right adjoint j. Given that j is right adjoint to a functor that acts as the identity map we have (N1) and (N2). In order to obtain (N3) we used the result mentioned above. Thus, we have $S = \mathcal{L}^j$. □

The following lemma is a corollary of the above characterization.

2.18 Lemma. *The intersection of fixed point sets of nuclei (as sets) is again a fixed point set.*

Thus arbitrary intersection of regular sublocales correspond to intersection as sets of the corresponding images of their fixed points sets. We can now define density as follows.

2.19 Definition. A regular sublocale $\mathcal{M}$ of $\mathcal{L}$ is a *dense sublocale*, iff the intersection of $\mathcal{M}$ with any open, non trivial sublocale of $\mathcal{L}$ is nonempty.

Is easy to obtain an alternative definition of density in terms of the associated nucleus.

2.20 Lemma. *Let k be a nucleus on $\mathcal{L}$. $\mathcal{L}^k$ is a dense sublocale of $\mathcal{L}$ iff $k(0) = 0$.*

Proof. Let $\mathcal{L}^k$ be a dense sublocale, if $k(0) = a$, we have $\mathcal{L}^{j_a} \cap \mathcal{L}^k = \{1\}$, since if $b \in \mathcal{L}^{j_a} \cap \mathcal{L}^k$, $b = k(0) \Rightarrow k(b) = 1$. So a must be 0.
Let's suppose that $k(0) = 0$ and let a be in $\mathcal{L}$. We have to show that $\neg a \in \mathcal{L}^k \cap \mathcal{L}^{j_a}$. We have $\neg a \in \mathcal{L}^k$ since

$$\neg a = a \Rightarrow 0 = a \Rightarrow k(0) = k(a) \Rightarrow k(0),$$

and so we can easily conclude $k(\neg a) \leq \neg a$, the other inequality is obtained directly. $\neg a \in \mathcal{L}^{j_a}$ follows from the laws of intuitionistic logic. If $a \neq 0$, then $\neg a \neq 1$. □

With the above proof in mind we get the following result.

2.21 Theorem. *Isbell's density theorem* [PP, p. 40]. *$\mathcal{L}^{\neg\neg}$ is dense, and it is the intersection of all dense sublocales of $\mathcal{L}$.*

2.22 Example.

- If $a \in \mathcal{L}$, then the open sublocale $\downarrow a$ is dense in $\mathcal{L}$ iff $a \Rightarrow 0 = \neg a = 0$, as in the topological case, this is equivalent to the assertion that, given $b \in \mathcal{L}$, if $b \wedge a = 0$, then $b = 0$. The closed sublocale $\uparrow a$ is dense iff $a \vee 0 = a = 0$.
- If $a \leq b \in \mathcal{L}$, then $\downarrow a$ is dense in $\downarrow b$ iff $\neg a \wedge b = 0$, as in the topological case this is equivalent to the fact that, for $c \leq b$, if $c \wedge a = 0$, then $c = 0$.
- If $a \in \mathcal{L}$, then the open sublocale $\downarrow (a \vee \neg a)$ is dense in $\mathcal{L}$ since $\neg(a \vee \neg a) = \neg a \wedge \neg\neg a = 0$.

Next we define an adjunction between the category of locales $\mathbf{Loc}$, and the category of topological spaces $\mathbf{Top}$ that will allow us to establish which topological spaces can be reconstructed from their lattices of open sets, and which locales can be obtained as the lattices of open sets of topological spaces. In this way we can give examples of locales that do not arise from topological spaces. We follow the detailed proof which is found in [Bo94, p. 61-73]. For the next definition, we note that $\{0, 1\}$ is the terminal object of the category of locales since, given a locale $\mathcal{L}$ the only morphism $f : \mathcal{L} \to \{0, 1\}$ is given by $f^*(0) = 0$ y $f^*(1) = 1$.

2.23 Definition. A *point of a locale* $\mathcal{L}$ is a morphism $p : \mathbf{1} \to \mathcal{L}$ of locales, where $\mathbf{1}$ is the terminal locale.

2.24 Example. Let $(X, \mathcal{T})$ be a topological space, for $x \in X$ we can define the point $p_x : \mathbf{1} \to \mathcal{T}$ such that $p_x^*(U) = 1$ iff $x \in U$.

2.25 Lemma. [Bo94, p. 62] *There is a bijective correspondence between points of a locale and elements $u \in \mathcal{L}$ which satisfy $u \neq 1$ and, for all $a, b \in \mathcal{L}$, if $a \wedge b \leq u$, then $a \leq u$ o $b \leq u$. The mentioned elements of $\mathcal{L}$ are called prime elements.*

Proof. Let $p : \mathbf{1} \to \mathcal{L}$, we define the prime element $u = \bigvee\{a : p^*(a) = 0\}$. If u is a prime element, we define a point p such that, for $a \leq u$, $p^*(a) = 0$, and $p^*(a) = 1$ in other case. These mappings are the inverse of each other. □

We associate a topological space to each locale in a natural fashion.

2.26 Lemma. [Bo94, p. 64] *Let $\mathcal{L}$ be a locale. We define a topology in the set of points of $\mathcal{L}$, which we denote by $\mathbf{Sp}(\mathcal{L})$ (the spectrum of $\mathcal{L}$). The open sets of this topology are given by $\mathcal{O}_a = \{p \in \mathbf{Sp}(\mathcal{L}) : p^*(a) = 1\}$, for $a \in \mathcal{L}$. That this sets constitute a topology can be inferred from the satisfaction of the following equalities:*

$$\mathcal{O}_0 = \emptyset,\ \mathcal{O}_1 = \mathbf{Sp}(\mathcal{L}),\ \mathcal{O}_{\bigvee a_i} = \bigcup \mathcal{O}_{a_i},\ \mathcal{O}_{a \wedge b} = \mathcal{O}_a \cap \mathcal{O}_b.$$

The above definition can be extended to a functor between $\mathbf{Loc}$ and $\mathbf{Top}$ which turns out to be the right adjoint of the forgetful functor.

2.27 Proposition. [Bo94, p. 65] *Consider the forgetful functor $\mathcal{O} : \mathbf{Top} \to \mathbf{Loc}$ which maps the space $(X, \mathcal{T})$ to its lattice of open sets $\mathcal{T}$ and a continuous function between topological spaces to its associated morphism of locales. This functor admits the spectrum functor $\mathbf{Sp} : \mathbf{Loc} \to \mathbf{Top}$ which maps $\mathcal{L}$ to the topological space $\mathbf{Sp}(\mathcal{L})$ as its right adjoint.*

Proof. We define the spectrum functor for morphisms as follows. If $f : \mathcal{L} \to \mathcal{M}$, $\mathbf{Sp}(f) : \mathbf{Sp}(\mathcal{L}) \to \mathbf{Sp}(\mathcal{M})$ maps p into $f \circ p$. The unity of the adjunction is defined for a topological space $(X, \mathcal{T})$ as the continuous function $\eta_{(X,\mathcal{T})} : X \to \mathbf{Sp}(\mathcal{T})$ which maps $x \mapsto p_x$. The counity define is defined for a locale $\mathcal{L}$ to be the morphism of locales $\varepsilon_{\mathcal{L}} : \mathcal{O}(\mathbf{Sp}(\mathcal{L})) \to \mathcal{L}$, which maps $\varepsilon^*_{\mathcal{L}}(a) = \mathcal{O}_a$. ☐

The former adjunction gives rise to an equivalence of categories if we ask that the corresponding unity and counity are isomorphisms. This leads us to the next definition.

2.28 Definition (Locale with enough points, sober topological space)**.**

- A locale $\mathcal{L}$ is a *locale with enough points* iff the morphism $\varepsilon_{\mathcal{L}}$ is an isomorphism.
- A topological space $(X, \mathcal{T})$ is *sober* iff the morphism $\eta_{(X,\mathcal{T})}$ is a homeomorphism.

By definition the morphism $\varepsilon^*_{\mathcal{L}}$ is surjective but it can be non injective. The morphism $\eta_{(X,\mathcal{T})}$ can be neither injective nor surjective. The spectrum of a locale is a sober topological space and if a topological space is Hausdorff, then it is sober. Besides, we have that if $(X, \mathcal{T})$ is a topological space, then $\mathcal{T}$ has enough points. Based on this we can give examples of locales that do not arise as the lattice of open sets of a topological space.

2.29 Example. Let $(X, \mathcal{T})$ be a Hausdorff topological space without isolated points, and let's consider the locale of regular elements of $\mathcal{T}$, $\mathcal{T}^{\neg\neg}$ the least dense sublocale. These elements constitute a Boolean algebra [Bo94, p. 11]. Let's see that the former algebra has no points. We can show this by showing that it has no prime elements. Let's suppose that there is a prime element $U \in \mathcal{T}^{\neg\neg}$, then U is not the top element. Since U is regular, U equals the interior of $\overline{U}$ and then $\overline{U} \neq X$. There is some $x \notin \overline{U}$, and some open set that contains x and do not intersect U, as there is no isolated points in X there is some other element $y \neq x$ in this open set. Since we are in a Hausdorff space we can choose open sets V and W such that $x \in V, y \in W$ and $V \cap W = \emptyset$. $\neg\neg V$ and $\neg\neg W$ are regular elements and

$$\neg\neg V \cap \neg\neg W = \neg\neg(V \cap W) = \neg\neg\emptyset = \emptyset \subseteq U.$$

And then, as U is a prime element, $\neg\neg V \subseteq U$ or $\neg\neg W \subseteq U$, and we have $x \in U$ or $y \in U$, which is impossible. As this locale has no points, it can not be the lattice of open sets of a topological space.

2.30 Theorem. [Bo94, p. 73] *The category of locales with enough points (and locale morphisms) is equivalent to the category of sober topological spaces.*

Proof. The spectrum of a locale is a sober topological space, and the lattice of open sets of a topological space is a locale with enough points, hence the adjunction between topological spaces and locales can be restricted to an adjunction between the full subcategory of sober topological spaces and the full subcategory of locales with enough points. This adjunction turns out to be an equivalence of categories since, by definition, the unities and counities of the adjunction are now isomorphisms. □

We can conclude also that if a locale $\mathcal{T}$ arises from a topological spaces then, as it has enough points, it is isomorphic to the lattice of open sets of some sober topological space.

Next we define connected a compact locales.

2.31 Definition (Compact locale, connected locale)**.**

- An element u of a locale $\mathcal{L}$ is said to be *connected* iff for all $w, v \in \mathcal{L}$, if $u = w \vee v$ and $w \wedge v = 0$, we have $w = 0$ or $v = 0$.
- A locale $\mathcal{L}$ is *connected* iff the element $1 \in \mathcal{L}$ is connected.
- Let a, b be elements of a locale $\mathcal{L}$, we say that $a << b$ *(a is way below b)* if for all family $(c_i)_{i \in I}$ of elements of $\mathcal{L}$ such that $b \leq \bigvee_{i \in I} c_i$, there is a finite subset $J \subseteq I$ such that $a \leq \bigvee_{j \in J} c_j$.
- An element $a \in \mathcal{L}$ is *compact* iff $a << a$. $\mathcal{L}$ is *compact* iff $1 \in \mathcal{L}$ is compact.

Chapter 3

Logic of sheaves of structures on a locale

Topological sheaves arose in Weyl's work on Riemann surfaces, even though the current definition is due to Leray and Cartan and their full development and application to Serre, Godement and Grothendieck. The work on topological sheaves led to the invention of Grothendieck toposes, which can be thought of as generalized sheaf categories, and elementary Lawvere toposes, which are a simplification of Grothendieck toposes. In elementary toposes it is possible to define some high order semantics, known as Kripke-Joyal, which turns out to satisfy the laws of intuitionistic logic. The laws that define this semantics can be obtained in a natural way from a pointwise logic in sheaves of first order structures [Ca95]. This pointwise logic can be used to force semantics on open sets in such a way that the epistemic principle of continuity of truth is satisfied, all statement that holds in a point must hold in a neighborhood of the point. This generalizes the behavior of mathematical assertions such as analyticity of a complex variable function in a point and equality between sections of a sheaf.

Logic of sheaves of first order structures on a locale would occupy an place between Caicedo's semantics for first order structures and Kripke-Joyal semantics. An epistemic motivation to study such logic can be found again in the principle of continuity of truth. Since the points are no more than ideal limits of observable surroundings, we ask: what happens if in order to define the semantics we forget about points from the beginning? In what follows we try to answer this question. Although we could not describe sheafification (the relation between categories of presheves and sheaves of structures) analogously to the case of sheaves of sets, via tale morphisms, we will describe it following the lines of Fourman and Scott's work [FS79], who studied a category which, under certain restrictions, turns out to be isomorphic to the one of sheaves of structures.

In [FS79], Fourman and Scott define the category of Ω-sets, where Ω is a locale and an Ω-set is set with a relation which mimics the behavior of equality between sections of the presheaf. They show that the category of Ω-sets is equivalent to the one of sheaves of sets on Ω and they describe in which way a presheaf of sets can be transformed in a sheaf by using the category of complete Ω-sets, a subcategory of Ω-sets isomorphic to the category of sheaves of structures. Although in [FS79] it is defined the concept of Ω-structure, there is no proof the equivalence between this category and the one of sheaves of structures, and even though it is evident that the category of sheaves of sets is reflective in the category of presheaves of sets, this is also not there.

In what follows it is shown, using the techniques in [FS79], that the category of Ω-structures, with some restrictions, is equivalent to the category of sheaves of structures on Ω and that the category of sheaves of structures on Ω is reflective in the category of presheaves of structures on Ω. As a corollary we obtain an isomorphism between the category of complete Ω-sets and the one of sheaves of structure on Ω. We will quote the proof of the isomorphism betweeen Ω-sets and sheaves of sets on Ω as it appears in [Bo94, p. 144-167], but we will eventually make references to [FS79].

3.1 Sheafification

In what follows we fix a locale Ω. First we define the concept of presheaves and sheaves, and establish a motivation for the concepts of Ω-set and Ω-structure.

1.1 Definition. A *preshaf of sets on* Ω is a contravariant functor $F : \Omega \to \mathbf{Set}$. Given $v \leq u$ in Ω, we let ρ^u_v denote $F(u \to v)$ and $x|_v$ denote $\rho^u_v(x)$ if there is no ambiguity.

The category of presheaves of sets on Ω is the category $\mathbf{Set}^{\Omega^{op}}$, the morphisms being natural transformations between functors.

1.2 Definition. Let F be a presheaf of sets on Ω and $(u_i)_{i\in I}$ be a family of elements of Ω. The family $(x_i \in F(u_i))_{i\in I}$ of the preasheaf F is said to be *compatible* when for $i, j \in I$,

$$x_i|_{u_i\wedge u_j} = x_j|_{u_i\wedge u_j}.$$

1.3 Definition. A *presheaf of sets* F on Ω is said to be *separated* when, given $u = \bigvee_{i\in I} u_i \in \Omega$ and $x, y \in F(u)$, if for all $i \in I$ $x|_{u_i} = y|_{u_i}$, then $x = y$.

1.4 Definition. A presheaf of sets F on a locale Ω is said to be a *sheaf* when, given $u = \vee_{i\in I} u_i \in \Omega$ and a compatible family $(x_i \in F(u_i))_{i\in I}$ of F, there is a unique $x \in F(u)$ such that, for all $i \in I$, $x|_{u_i} = x_i$. The element x will be called the gluing of the family $(x_i)_{i\in I}$.

From now on, we use τ to denote a first order language.

1.5 Definition. A *preshaf of structures of signature τ on Ω* is a contravariant functor $F : \Omega \to St_\tau$ from the locale Ω into the category of first order structures of signature τ whose morphisms are homomorphism of structures.

From now on we will use the notation $\bar{a} = (a_1, \ldots, a_n)$ to denote tuples of elements, as well as $\bar{b}, \bar{c}$, etc. The following definition is analogous to the definition of sheaves of structures of first order on topological spaces given in [Ca95].

1.6 Definition. A *sheaf of structures of signature τ on Ω* is a presheaf of structures on Ω, F, which when considered as presheaf of sets is a sheaf and such that given a symbol of relation R of arity n in τ, $u = \bigvee_{i\in I} u_i \in \Omega$ and $\bar{a} \in F(u)^n$, if for all $i \in I$ $R^{F(u_i)}(a_1|_{u_i}, \ldots, a_n|_{u_i})$, then $R^{F(u)}(\bar{a})$.

The category of sheaves of structures on Ω is the full subcategory of ${\mathbf{St}_\tau}^{\Omega^{op}}$ whose objects are sheaves of structures.

1.7 Example. In the examples of sheaves that we will study the locale will always come from topological spaces since we did not find interesting examples of sheaves on locales without points. However, to study these examples we will often use techniques coming from locale theory when we find it convenient.

- Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ topological spaces. We define, for $u \in \mathcal{T}$, $F(u) = \{f \mid f : u \to Y, f \text{ continuous}\}$, with the usual restrictions $\rho^u_v(f) = f|_v$, if $v \subseteq u$. This presheaves are evidently sheaves of sets. Depending on the topological space we can ask that the functions be more than continuous, smooth, holomorphic, etc.

- A case specially interesting of the former example is the sheaf of rings of germs of holomorphic functions G s.t. $G(U) = (\{g : U \to \mathbb{C} \mid g \text{ is holomorphic in } U\}, +, .)$, for $U \subseteq \mathbb{C}$, where the sum and product are defined pointwise.

- Let $p : (Y, \mathcal{S}) \to (X, \mathcal{T})$ a local homomorphism, for $U \in \mathcal{T}$ let

  $$G(U) = \{s \mid U \to Y : s \text{ is continuous and } p \circ s = id_U\},$$

  then G is a sheaf, known as the sheaf of continuous sections of p. The above presentation of sheaves is called topological sheaf or tale space. This example is important because one can prove that every sheaf (of sets or structures) on a topological space is isomorphic to a sheaf of this kind. Furthermore, this construction allows us to describe the sheafification functor in the topological case. The case of sheaves of sets is explained in detail in [Bo94, p. 113-123]. In the case of sheaves of structures, it must be asked that for $x \in X$, the fiber $p^{-1}(x)$ be a first order structure of signature τ, so that $G(U)$ becomes a structure of signature τ as substructure of the product $\prod_{x\in U} p^{-1}(x)$. Additional conditions of continuity must be asked for the interpretation of functions and relations [Ca95, p. 10].

- A specially interesting case of the former example is the sheafification of a ring. Let R be a commutative ring with unity and consider its set of prime ideals $Sp(R)$, the spectrum of R, with the Zariski topology generated by basic open sets

$$\mathcal{O}_a = \{J \mid J \text{ prime ideal of } R \text{ and } a \notin J\},$$

for $a \in R$. $Sp(R)$ is a sober topological space and each $\mathcal{O}_a$ turns out to be compact [Bo94, p. 175]. For every prime ideal J we can consider the ring R localized in J, $R_J = R(R \setminus J)^{-1}$ [Bo94, p. 173] and define the topological sheaf

$$\pi : \coprod_{J\in Sp(R)} R_J \to Sp(R), x \in R_J \to J,$$

where the topology of $\coprod_{J\in Sp(R)} R_J$ is the final topology induced by sections

$$s_b^a : \mathcal{O}_b \to \coprod_{J\in Sp(R)} R_J, J \to \frac{a}{b} \in R_J$$

for $a, b \in R$. This tale space is known as the structural space of R. The sheaf of continuous sections of this structural space of R [Bo95, p. 177] given by π is in fact a sheaf of rings [Bo95, p. 178]. It is worth to note that interpretations of the constants 0 y 1 are sections s_1^0 y s_1^1 respectively.

The category of Ω-sets implies a slight change of view. Instead of considering the sheaf as a whole we consider a generating subset.

1.8 Definition. [Bo94, p. 138] Let F be a sheaf of sets on Ω. We say that a family, A, of elements of $\coprod_{u\in\Omega} F(u)$ is a *family of generators forF* if for all $u \in \Omega$ and $a \in F(u)$, there is a covering $u = \bigvee_{i\in I} u_i$ and elements $a_i \in A$, such that $a|_{u_i} = a_i|_{u_i}$.

1.9 Example. If F is a sheaf of sets on Ω and $\{u\}_{u\in B}$ is a base for Ω (that is to say $B \subseteq \Omega$ and for $u \in \Omega$ there is $\{u_i\}_{i\in I}$, $I \subseteq B$ such that $u = \bigvee_{i\in I} u_i$), then $\coprod_{u\in B} F(u)$ is a family of generators for F.

A family of generators, A, allows us to reconstruct the sheaf in a unique way. If an element does not belong to A it will be at least a gluing of some restrictions of elements of A. All we need to know in order to reconstruct this element is how does it compare to all individuals in A. With the purpose of coding this information in mind we introduce a relation of comparison between elements of a sheaf which defines the maximum level in which the restriction of the elements are equal.

1.10 Definition. [Bo94, p. 139] Let F be a sheaf on Ω. Given $u, v \in \Omega$, $a \in F(u)$ and $b \in F(v)$ we define

$$[a \approx b] = \bigvee \{w \in \Omega \mid w \leq u \wedge v, a|_w = b|_w\},$$

where the join is a maximum since F is a sheaf.

We will be interested in extracting the essential properties of the relation $\approx$, which will be used to define the objects in the category of Ω-sets. These are sets with an equality which takes values in Ω instead of $\{0, 1\}$.

1.11 Lemma. [Bo94, p. 139] *Let F be a sheaf of sets on a locale Ω and A be a family of generators for F. For all $a, b, c \in A$ we have:*

- $[a \approx b] = [b \approx a]$
- $[a \approx b] \wedge [b \approx c] \leq [a \approx c]$

In this way, we can see how sheaves of sets can be regarded as sets with an equality which takes values in Ω. Morphisms between sheaves can also be described using the information coming from the information of each morphism in relation to a generating family. However, a morphism between sheaves does not necessarily defines a function between generating subsets since the image of some element might not belong to the generating subset of the target sheaf, however it must be a gluing of the generating subset. In order to reconstruct the function we need to compare the image of an element with all the elements in the generating family of the target sheaf. The following lemma motivates the definition of morphisms in the category of Ω-sets.

1.12 Lemma. [Bo94, p. 140] *Let A and B be families of generators of the sheaves of sets$F$ and G over Ω respectively. Let $\alpha : F \to G$ be a morphism of sheaves where for all $u \in \Omega$, $\alpha_u : F(u) \to G(u)$ is the level u of the morphism α. We define the function*

$$A \times B \to \Omega, \ (a, b) \mapsto [b \sim \alpha(a)] := [b \approx \alpha_u(a)]$$

for all $a \in F(u)$. For $a, a' \in A$ and $b, b' \in B$ the following relations hold:

(1) $[b \approx b'] \wedge [b' \sim \alpha(a)] \leq [b \sim \alpha(a)]$.

(2) $[b \sim \alpha(a)] \wedge [a \approx a'] \leq [b \sim \alpha(a')]$.

(3) $[b \sim \alpha(a)] \wedge [b' \sim \alpha(a)] \leq [b \approx b']$.

(4) $[a \approx a] = \bigvee_{b \in B} [b \sim \alpha(a)]$.

In the case where A and B are not just families of generators but the whole sheaf, the fist three conditions imply that α is a natural transformation, while the fourth indicates that the image of an element $a \in F(u)$ belongs to $G(u)$. As we will show later, the relation $[b \sim \alpha(a)]$ encodes all the information of morphism α, giving descriptions of each $\alpha(a)$. The following lemma justifies the definition of the composition of morphisms in the category of Ω-sets.

1.13 Proposition. [Bo94, p. 140] *Let A, B, C be families of generators of the sheaves of sets on Ω F, G and H respectively. Let $\alpha : F \to G$ and $\beta : G \to H$ be two morphisms of sheaves, using the previous notation, we have*

$$[c \sim \beta \circ \alpha(a)] = \bigvee_{b \in B} [c \sim \beta(b)] \wedge [b \sim \alpha(a)]$$

for all $a \in A$ and $c \in C$.

Now we define the elements of the category of Ω-sets, our examples being families of generators of sheaves of sets.

1.14 Definition. [Bo94, p. 144] An Ω-*set* is a pair $(A, \approx)$, where A is a set and $\approx$ is a function, also called Ω-equality

$$\approx : A \times A \to \Omega, \ (a, b \mapsto [a \approx b]),$$

which satisfies the following for all $a, b, c \in A$:

(C1) $[a \approx b] = [b \approx a]$.

(C2) $[a \approx b] \wedge [b \approx c] \leq [a \approx c]$.

The following technical lemma will be used often.

1.15 Lemma. [Bo94, p. 144] *Let A be an Ω-set. For all $a, b \in A$ the following relation holds*

$$[a \approx b] \leq [a \approx a].$$

Next we define morphisms between Ω-sets. This will allow us to define the category of Ω-sets. Our examples again are morphisms between families of generators of sheaves of sets.

1.16 Definition. [Bo94, p. 145] Let A, B be Ω-sets. A *morphism of* Ω*-sets* $\alpha : A \to B$ is a mapping

$$A \times B \to \Omega,\ (a, b) \mapsto [b \sim \alpha(a)],$$

which satisfies the following for all $a, a' \in A$ and $b, b' \in B$:

(M1) $[b \approx b'] \wedge [b' \sim \alpha(a)] \leq [b \sim \alpha(a)]$.

(M2) $[b \sim \alpha(a)] \wedge [a \approx a'] \leq [b \sim \alpha(a')]$.

(M3) $[b \sim \alpha(a)] \wedge [b' \sim \alpha(a)] \leq [b \approx b']$.

(M4) $[a \approx a] = \bigvee_{b \in B}[b \sim \alpha(a)]$.

The following technical lemma will be used often.

1.17 Lemma. [Bo94, p. 145] *Let* $\alpha : A \to B$ *a morphism of* Ω*-sets. If* $a \in A$ *and* $b \in B$*,* $[b \sim \alpha(a)] \leq [a \approx a] \wedge [b \approx b]$.

Now we can define the category of Ω-sets.

1.18 Proposition. [Bo94, p. 145] *Let* Ω *be a locale.* Ω*-sets and morphisms of* Ω*-sets constitute a category if we define composition for* $\alpha : A \to B$ *and* $\beta : B \to C$ *by*

$$[c \sim \beta \circ \alpha(a)] = \bigvee_{b \in B} [c \sim \beta(b)] \wedge [b \sim \alpha(a)]$$

for all $a \in A$ *and* $c \in C$*. The identity* $1_A : A \to A$ *of an* Ω*-set* A *is given by*

$$[a \sim 1_A(a')] = [a \approx a'],$$

for $a, a' \in A$*. We denote this category by* Ω-$\mathbf{Set}$.

1.19 Example. If A is a set, consider the $\{0, 1\}$-equality given by $[a \approx b] = 0$ iff $a \neq b$. This is a $\{0, 1\}$-set, (C1) and (C2) reduced to reflexivity and transitivity of equality. $\mathbf{Set}$ turns out to be equivalent to the category of $\{0, 1\}$-sets.

In order to work with morphisms of Ω-sets we prove the following lemma. If A, B are sets and $f, g : A \to B$ are functions we have that $f = g$ iff for all $b \in B$ and $a \in A$, if $b = f(a)$ then $b = g(a)$. The next lemma mimics the former assertion.

1.20 Lemma. [Bo94, p. 147] *Let Ω be a locale and $\alpha, \beta : A \to B$ morphisms of Ω-sets. The following assertions are equivalent:*

(1) $\alpha = \beta$

(2) For all $a \in A$ and $b \in B$ we have $[b \sim \alpha(a)] \leq [b \sim \beta(a)]$.

We will consider the category of Ω-sets and Ω-structures as accessories in the proof of the existence of a sheafification functor (which describes the relation between the categories of sheaves and presheaves of structures). The concept of complete Ω-set, equivalent to that of sheaf of sets, plays an important role in the description of the sheafification functor. In order to define it we will use the concept of singleton. Let F be a sheaf and A be a family of generators, we have said that in order to recover an element $m \in \coprod_{u\in\Omega} F(u)$ we just need to know how does m compare with the rest of the elements in A, that is $\{[m \approx a]\}_{a\in A}$. The idea behind the definition of singleton is that each of these should be a coherent description of an element that may or may not be in A. Each family of the form $\{[m \approx a]\}_{a\in A}$ is one of these descriptions.

1.21 Definition. Let A be an Ω-set. A *singleton of* A is a mapping $\sigma : A \to \Omega$ such that, for all $a, b \in A$ the following holds:

(1) $\sigma(a) \wedge \sigma(b) \leq [a \approx b]$,

(2) $[a \approx b] \wedge \sigma(b) \leq \sigma(a)$.

As we have said, each element in an Ω-set induces one of these descriptions.

1.22 Lemma. *Let A be an Ω-set and $m \in A$. The mapping $\sigma_m : A \to \Omega$, defined by $\sigma_m(a) = [m \approx a]$, for all $a \in A$, is a singleton.*

In fact, if F is a sheaf, A a family of generators and $m \in \coprod_{u\in\Omega} F(u)$, then $\sigma_m : A \to A$ defined by $a \mapsto [m \approx a]$ is still a singleton. The following lemma, besides proving that any presheaf is naturally an Ω-set, allows us to give an alternative characterization of separated presheaf and sheaf in terms of the mapping $m \mapsto \sigma_m$.

1.23 Lemma. *Let F be a presheaf of sets on Ω. The set $A = \coprod_{u\in\Omega} F(u)$ is an Ω-set when endowed with the Ω-equality defined by*

$$[a \approx b] = \bigvee\{w \in \Omega : a|_w = b|_w\},$$

for all $a, b \in A$. For the mapping $\sigma : A \to \sigma(A)$ we have:

- *σ is injective iff F is a separated presheaf.*

- *σ is bijective iff F is a sheaf.*

The first assertion is proved in [Si01, p. 38]. The proof for the second assertion is ours, although in [Bo94, p. 142] the implication $\Leftarrow$ is proved with a little mistake that we amend here.

Proof. Let's see that the Ω-equality defines in fact an Ω-set. Symmetry is evident. For transitivity let a, b, d be elements of A. Let's denote $I = \{w \in \Omega : a|w = b|w\}$, $J = \{w \in \Omega : b|w = d|w\}$ y $K = \{w \in \Omega : a|w = d|w\}$, then we have

$$\begin{aligned}
[a \approx b] \wedge [b \approx d] &= \Big(\bigvee_{w\in I}\Big) \wedge \Big(\bigvee_{u\in J} u\Big) \\
&= \bigvee_{u\in J}\bigvee_{w\in I} w \wedge u \\
&\leq \bigvee_{v\in K} v \\
&= [a \approx d],
\end{aligned}$$

since if $w \in I$ and $u \in J$, then $a|w = b|w$ y $b|v = d|v$, so $a|_{w\wedge v} = (a|_w)|_{w\wedge v} = (b|_w)|_{w\wedge v} = (b|_v)|_{w\wedge v} = (d|_v)|_{w\wedge v} = d|_{w\wedge v}$. For $a \in F(u)$, we have $[a \approx a] = u$.

We show that σ is bijective iff F is a sheaf. Let's assume that σ is bijective and let $a_i \in F(u_i)$, for $i \in I$, be a compatible family and $u = \bigvee_{i\in I} u_i$. We define the mapping

$$\begin{aligned}
\rho : A &\to \Omega \\
a &\mapsto \bigvee_{i\in I}[a \approx a_i].
\end{aligned}$$

The fact that ρ is a singleton follows from a reasoning similar to the used above. By surjectivity there is $m \in A$ such that $\rho = \sigma_m$. For $i \in I$ we have

$$[m \approx a_i] = \sigma_m(a_i) = \rho(a_i) = \bigvee_{j\in I}[a_i \approx a_j] = [a_i \approx a_i] = u_i,$$

so that

$$[m \approx m] = \sigma_m(m) = \rho(m) = \bigvee_{i\in I}[m \approx a_i] = \bigvee_{i\in I} u_i = u,$$

this means that $m \in F(u)$. Let i be an element of I, using the presheaf separation we obtain $m|_{u_i} = a_i$. We have uniqueness since F is separated, being σ injective.

Assume now that F is a sheaf. Then F is separated and we have the injectivity of σ. Let ρ be a singleton, we want to show that there is $m \in F$ such that $\sigma_m = \rho$. Let's denote $u = \bigvee_{a \in A} \rho(a)$. Let$a$ be an element of A, since $\rho(a) = \rho(a) \wedge \rho(a) \leq [a \approx a]$ we can define the family $(a|_{\rho(a)} \in F(\rho(a)))_{a \in A}$, that turns out to be compatible since F is separated. Hence there is a (unique) gluing $m \in F(u)$ which satisfies, for $a \in A$, $m|_{\rho(a)} = a|_{\rho(a)}$.

Let a be in A, we have that $\rho(a) \leq [m \approx a]$. By using the definition of singleton we get $\rho(a) = [a \approx m] \wedge \rho(a) \leq \rho(m)$, so that $\bigvee_{a \in A} \rho(a) = \rho(m) = u$. In conclusion we note that

$$[m \approx a] = [a \approx m] \wedge [m \approx m] = [a \approx m] \wedge u = [a \approx m] \wedge \rho(m) \leq \rho(a).$$

Hence $\sigma_m = \rho$. □

In a separated sheaf, two elements with the same description are equal and in any sheaf, each description gives rise to an element in a unique way. This motivates the definition of complete Ω-set that will be equivalent that of sheaf of sets on Ω. The above lemma tells us half of it. Restrictions and levels will be defined in all complete Ω-sets, but this will be done more generally for the case of Ω-structures.

1.24 Definition. Let A be an Ω-set. Let $\sigma(A)$ be the set of singletons of A. We say that A is a *complete* Ω*-set* when the mapping

$$A \to \sigma(A),\ m \mapsto \sigma_m$$

is bijective.

The construction of the sheafification functor for presheaves of structures uses the category of complete Ω-structures (in analogy with Ω-sets) and of complete Ω-structures (in analogy with complete Ω-sets). In order to define the closest sheaf to a presheaf of structures we assing an Ω-structure to it. This Ω-structure will be isomorphic to some complete Ω-structure, and this will be naturally a sheaf of structures. Each of these transformations will be given by a functor. With the former sketch in mind we will go through the technical details of the definition of the category of Ω-stuctures.

In order to define Ω-structures we need interpretations of functions symbols as is the case for first order structures. The definition given in [FS79] requires that the interpretation of a function symbol be a mapping from A^n (set product) into A, which will contradict our motivation since families of generators of a sheaf won't be natural examples (if A is a family of generators, it is not necessary that the image of all elements in A^n be in A). With the goal of generalizing our definition of Ω-set and to be faithful to our motivation, the same of Fourman and Scott, that is, that families of generators be natural examples,we define the product in the category of Ω-sets, and let interpretations of function symbols be morphisms of Ω-sets rather than mappings of sets. This definition of product may result strange for a reader familiar with sheaf theory. When we define complete Ω-structures, we will be interested in defining the product in a more usual way.

1.25 Lemma. *Let A be an Ω-set. The set A^n, endowed with the Ω-equality given by*

$$[(\bar{a}) \approx (\bar{b})] = \bigwedge_{i=1}^{n} [a_i \approx b_i],$$

for all $\bar{a}, \bar{b} \in A^n$ is an Ω-set and is the product of n-times A in the category of Ω-$\mathbf{Set}$.

Proof. That A^n with the defined Ω-equality is an Ω-set follows from the fact that A is an Ω-set. Canonical projections $\pi_i : A^n \to A$ are defined, for all $c, a_1, \ldots, a_n \in A$ and $1 \leq i \leq n$, by the equation

$$[c \sim \pi_i(\bar{a})] = [c \approx a_i] \wedge [\bar{a} \approx \bar{a}].$$

Applying transitivity and the fact that the identity is a morphism of the category we have that each π_i is an Ω-morphism.

If B is an Ω-set, and $p_i : B \to A$ Ω-morphisms for $1 \leq i \leq n$, we can define $p : B \to A^n$ such that for all $\bar{a} \in A^n$ and $b \in B$,

$$[\bar{a} \sim p(b)] = \bigwedge_{i=1}^{n} [a_i \sim p_i(b)].$$

To see that p is an Ω-morphism we note that (M1)-(M3) follow from the fact that each p_i is a morphism of Ω-sets. For (M4), let b be an element of B, then

$$\begin{aligned}
\bigvee_{\bar{a} \in A^n} [\bar{a} \sim p(b)] &= \bigvee_{\bar{a} \in A^n} \left(\bigwedge_{i=1}^{n} [a_i \sim p_i(b)] \right) \\
&= \left(\bigvee_{a_1 \in A} [a_1 \sim p_1(b)] \right) \wedge \cdots \wedge \left(\bigvee_{a_n \in A} [a_n \sim p_n(b)] \right) \\
&= [b \approx b] \wedge \cdots \wedge [b \approx b] \\
&= [b \approx b].
\end{aligned}$$

The fact that for $1 \leq j \leq n$, we have $\pi_j \circ p = p_j$ follows by an easy calculation. Let $p^* : B \to A^n$ be a morphism of Ω-sets such that for all $1 \leq i \leq n$, $\pi_i \circ p^* = p_i$, we will show $p = p^*$.

Is easy to see that if $\bar{a} \in A^n$ and $b \in B$, $[\bar{a} \sim p(b)] \leq [\bar{a} \sim p^*(b)]$ and by the equality between Ω-morphisms lemma we have $p = p^*$. $\square$

If $f : A \to A$ is a mapping of sets, the notion of $(f, \ldots, f) : A^n \to A^n$ plays a role in the definition of first order structure, we need an analog in the category of Ω-sets.

1.26 Lemma. *Let A and B be Ω-sets and $\alpha : A \to B$ an Ω-morphism. If $\pi_i^A : A^n \to A$, $\pi_i^B : B^n \to B$ denote the i-th projection of the corresponding product for $1 \leq i \leq n$, then there is a unique Ω-morphism $(\alpha, \ldots, \alpha) : A^n \to B^n$ such that for $1 \leq i \leq n$ $\pi_i^B \circ (\alpha, \ldots, \alpha) = \alpha \circ \pi_i^A$.*

$$\begin{array}{ccc} A^n & \xrightarrow{(\alpha,\ldots,\alpha)} & B^n \\ {\scriptstyle \pi_i^A}\downarrow & & \downarrow{\scriptstyle \pi_i^B} \\ A & \xrightarrow{\alpha} & B \end{array}$$

Proof. This is evident from the fact that B^n is the product of n times B. If $a_i \in A$ and $b_i \in B$, for all $1 \leq i \leq n$ we have

$$[(\bar{b}) \sim (\alpha, \ldots, \alpha)(\bar{a})] = \bigwedge_{i=1}^{n} [b_i \sim \alpha(a_i)].$$

$\square$

1.27 Lemma. *With the notation of the above lemma, if $\alpha : A \to B$ and $\beta : B \to C$ are morphisms of Ω-sets, then $(\beta \circ \alpha, \ldots, \beta \circ \alpha) = (\beta, \ldots, \beta) \circ (\alpha, \ldots, \alpha)$ as follow by uniqueness of $(\alpha, \ldots, \alpha)$ and the commutativity of the following diagram.*

$$\begin{array}{ccccc} A^n & \xrightarrow{(\alpha,\ldots,\alpha)} & B^n & \xrightarrow{(\beta,\ldots,\beta)} & C^n \\ {\scriptstyle \pi_i^A}\downarrow & & {\scriptstyle \pi_i^B}\downarrow & & \downarrow{\scriptstyle \pi_i^C} \\ A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & c \end{array}$$

In order to define interpretations of constant symbols in Ω-structures we need a morphism from the terminal Ω-set into the Ω-structure universe. This morphisms are a particular case of descriptions of elements or singletons.

1.28 Lemma. *In the category of Ω-sets, the terminal object is the set $\{*\}$ (singleton) endowed with the Ω-equality $[* \approx *] = 1$.*

Proof. If A is an $\Omega-$ set, we can define the morphism $\alpha : A \to \{*\}$ by the equation $[* \sim \alpha(a)] = [a \approx a]$. Satisfaction of (M1)-(M3) follows from the features of A's Ω-equality and $[* \approx *] = 1$. (M4) is satisfied trivially and guarantees the uniqueness of α. $\square$

The following definition is necessary for interpreting constant symbols in Ω-structures.

1.29 Definition. A *point* of an Ω-set A is a morphism $\alpha : \{*\} \to A$.

Points are no more that singletons which describe globally defined elements, in sheaves of structures these will define global sections as in [Ca95].

1.30 Lemma. *Let A be an Ω-set. The points of A are in bijective correspondence with the singletons σ of A which satisfy $\bigvee_{a\in A} \sigma(a) = 1$.*

Proof. If $\alpha : \{*\} \to A$ is a point of A, we can define the singleton $\sigma_\alpha : A \to \Omega$ such that, for all $a \in A$ $\sigma_\alpha(a) = [a \sim \alpha(*)]$. Axioms for singleton are satisfied and

$$\bigvee_{a\in A} \sigma_\alpha(a) = \bigvee_{a\in A} [a \sim \alpha(*)] = [* \approx *] = 1.$$

On the other hand, if σ is a singleton such that $\bigvee_{a\in A} \sigma(a) = 1$ we can define $\alpha_\sigma : \{*\} \to A$ by the equation $[a \sim \alpha_\sigma(*)] = \sigma(a)$. (M1) to (M3) for α_σ follows from the fact that σ is a singleton and (M4) follows from $\bigvee_{a\in A} \sigma(a) = 1$. The above mappings between points and singletons σ such that $\bigvee_{a\in A} \sigma(a) = 1$ are the inverse of each other and hence bijective. □

We are ready to define Ω-structures. Our definition will differ from the one in [FS79] mainly in that we won't ask interpretation of function and constant symbols to be functions and globally defined elements(i.e. such that $[a \approx a] = 1$) of the Ω-sets but rather descriptions of functions (Ω-morphisms) and global descriptions of elements (points), being consistent with our motivation of considering families of generators instead of preshaves as our main example. In [FS79] it is defined the following notion:

1.31 Definition. [FS79, p. 341]Given A an Ω-set, we define for all $a, b \in A$,

$$\begin{aligned}[][a \equiv b] &= ([a \approx a] \Rightarrow [a \approx b]) \wedge ([b \approx b] \Rightarrow [a \approx b]) \\ &= \bigvee\{c : c \wedge [a \approx a] \wedge [b \approx b] \leq [a \approx b]\}\end{aligned}$$

We say that an element of an Ω-set A, a, is defined in $u \in \Omega$ if $u \leq [a \approx a]$. The interpretations of functions and relations in [FS79], more than preserving the notion of $\approx$, as in our case, preserve the notion $\equiv$. The difference is that, for example in the case of presheaves, a and b turn out to be equivalent not only where its restriction coincides but also where non of them is defined. Following [Ca95] we just care about what happens when a and b are defined. The above definition is just there to highlight the difference between the definition of Ω-structures in [FS79] and ours. Another difference is that we ask, under the same motivation, the interpretations of relations to be restricted.

1.32 Definition. Let τ be a first order language. An Ω-*structure*, $\mathfrak{A}$, of signature τ consists of:

- An Ω-set A.
- For each constant symbol c of τ a point $c^{\mathfrak{A}} : \{*\} \to A$.
- For each function symbol f of arity n an Ω-morphism $f^{\mathfrak{A}} : A^n \to A$.
- For each relation symbol R of arity n a mapping $R^{\mathfrak{A}} : A^n \to \Omega$, where if $\bar{a} \in A^n$, we denote $R^{\mathfrak{A}}(\bar{a})$ by $[R^{\mathfrak{A}}(a_1, \dots, a_n)]$. This mapping must satisfy for all $1 \leq i \leq n$ and $a_i, b_i \in A$:

  [Extensionality] $\bigwedge_{i=1}^{n}[a_i \approx b_i] \wedge [R^A(\bar{a})] \leq [R^A(\bar{b})]$.

  [Restriction] $[R^A(a_1, \dots, a_n)] \leq \bigwedge_{i=1}^{n}[a_i \approx a_i]$.

Sometimes we will use c^A, f^A and R^A when there is no possible misunderstanding. Since $\{*\}$ is the terminal object of the category Ω-$\mathbf{Set}$ we Can consider constant symbols as functional symbols of arity 0 as in the classic case. From now on we will use n to denote the arity of function and relation symbols of τ, when there is no confusion.

1.33 Definition. If $\mathfrak{A}, \mathfrak{B}$ are Ω-structures of signature τ, a *morphism of* Ω-*structures* $\alpha : \mathfrak{A} \to \mathfrak{B}$ is a morphism of Ω-sets $\alpha : A \to B$ which satisfies:

- For each constant symbol c in τ $\alpha \circ c^A = c^B$.
- For each function symbol f in τ $\alpha \circ f^A = f^B \circ (\alpha, \dots, \alpha)$.

$$\begin{array}{ccc} A^n & \xrightarrow{(\alpha,\dots,\alpha)} & B^n \\ {\scriptstyle f^A}\downarrow & & \downarrow{\scriptstyle f^B} \\ A & \xrightarrow[\alpha]{} & B \end{array}$$

- For each relation symbol R in τ, $\bar{a} \in A^n$ and $\bar{b} \in B^n$,

$$[R^A(\bar{a})] \wedge [(\bar{b}) \sim (\alpha, \dots, \alpha)(\bar{a})] \leq [R^B(\bar{b})].$$

1.34 Lemma. *Let* $\alpha : \mathfrak{A} \to \mathfrak{B}$, $\beta : \mathfrak{B} \to \mathfrak{C}$ *be morphisms of* Ω-*structures for* τ, *the morphism of* Ω-*sets* $\beta \circ \alpha : A \to C$ *is in fact a morphism of* Ω-*structures.* $1_A : A \to A$ *(the identity of* Ω-*sets) is also a morphism of* Ω-*structures.*

Proof. Let's see that $\beta \circ \alpha$ is a morphism of Ω-structures. If c is a constant symbol, $\beta \circ \alpha \circ c^A = \beta \circ c^B = c^C$. The condition for function symbols is obtained by lemma **1.27**. The condition for relation symbols follows by an easy calculation.

Let's show now that 1_A is also a morphism of Ω-structures. The condition for constant symbols is analog to the case of function symbols, so it follows since $(1_A, \ldots, 1_A) = 1_{A^n}$. The condition for relation symbols is extensionality. $\square$

The above lemma allows us to define Ω-structures as a category.

1.35 Definition. The *category of* Ω*-structures* for a first order language τ is defined by the former lemma. This category will be denoted by Ω-St_τ.

We have seen that any sheaf of sets is naturally a complete Ω-set. In the same way sheaves of structures will be complete Ω-structures. In fact, the category of complete Ω-structures is isomorphic to the category of sheaves of structures.

1.36 Definition. We say that an Ω-structure for τ, $\mathfrak{A}$, is complete when it is complete as an Ω-set.

The following lemma says that any Ω-structure can be naturally completed and hence that any presheaf can be turn into a sheaf. In [FS79, p. 360] there is a proof for a restricted case of the definition in objects although, since our definition is different, the proof does not cover our case, which has never been treated before. The case where $\tau = \emptyset$, that is to say, the case of Ω-sets, can be found proven in detail in [Bo94, p. 157]. Basically, given an Ω-structure A, we consider the family of singletons of A which can be seen naturally as an Ω-structure. We can get an intuition of all the definitions if we think in the case where A is a family of generators of a sheaf of sets F, singletons in this case are descriptions of elements of the sheaf. For example, for the definition of the Ω-equality between singletons, if ρ is a singleton, $\rho(a)$ is the maximum level in which the restriction of the element described by ρ, m, is equal to a. If χ is another singleton, the element described by ρ and the one described by χ, n, must coincide at least in $\rho(a) \wedge \chi(a)$. On the other hand, since m and n are gluings of elements in A, we must have $[m \approx n] = \bigvee_{a \in A} \rho(a) \wedge \chi(a)$, and so, this must be the maximum level in which ρ and χ coincide. Definitions of interpretations of function, constant and relation symbols admit a similar motivation. The advantage of considering the Ω-structure$\sigma(A)$ lies in its completeness (that is, it will be naturally a sheaf of structures) and in the essential fact that, as an Ω-structure, it is isomorphic to A.

1.37 Lemma. *We define a functor* σ*, from the category of* Ω*-structures to its full subcategory whose objects are complete* Ω*-structures. Let* $\mathfrak{A}$ *be an* Ω*-structure for* τ*. The set of singletons of* A, $\sigma(A)$, *is a complete Ω-set with the Ω-equality defined by*

$$[\rho \approx \chi] = \bigvee_{a \in A} \rho(a) \wedge \chi(a)$$

for all $\rho, \chi \in \sigma(A)$. Besides, it is a complete Ω-structure if:

- *For each constant symbol c in τ, $c^{\sigma(\mathfrak{A})} : \{*\} \to \sigma(A)$ is given by the equation*

  $$[\rho \sim c^{\sigma(\mathfrak{A})}(*)] = \bigvee_{a \in A} \rho(a) \wedge [a \sim c^{\mathfrak{A}}(*)],$$

  for all $\rho \in \sigma(A)$.

- *For each function symbol in τ, $f^{\sigma(A)} : \sigma(A)^n \to \sigma(A)$ is given by the equation*

  $$[\rho \sim f^{\sigma(\mathfrak{A})}(\bar{\chi})] = \bigvee_{b, a_1, \ldots, a_n \in A} [b \sim f^{\mathfrak{A}}(\bar{a})] \wedge \rho(b) \wedge \bigwedge_{i=1}^{n} \chi_i(a_i),$$

  for all $\rho, \chi_1, \ldots, \chi_n \in \sigma(A)$.

- *For each relation symbol R in τ*

  $$[R^{\sigma(\mathfrak{A})}(\bar{\rho})] = \bigvee_{\bar{a} \in A^n} [R^{\mathfrak{A}}(\bar{a})] \wedge \bigwedge_{i=1}^{n} \rho_i(a_i),$$

for all $\bar{\rho} \in \sigma(A)^n$. Let $\mathfrak{B}$ be a complete Ω-structure of signature τ and let $\alpha : \mathfrak{A} \to \mathfrak{B}$ be a morphism of Ω-structures. Then $\sigma(\alpha) : \sigma(\mathfrak{A}) \to \sigma(\mathfrak{B})$ defined by the equation

$$[\chi \sim \sigma(\alpha)(\rho)] = \bigvee_{a \in A, b \in B} [b \sim \alpha(a)] \wedge \rho(a) \wedge \chi(b),$$

for $\chi \in \sigma(B)$ and $\rho \in \sigma(A)$, is a morphism of Ω-structures.

Proof. We will assume that $\sigma(A)$ is a complete Ω-set with the Ω-equality defined, as it is proved in [Bo94, p. 157]. Let f be a function symbol in τ, we will show that f^A is an Ω-morphism. (M1), (M2) and (M3) follows from the definition of singleton, and the fact that f^A is morphism of Ω-sets.

For (M4), let $\bar{\chi}$ be an element of $\sigma(A)^n$, then

$$
\begin{aligned}
\bigvee_{\nu\in\sigma(A)} [\nu \sim f^{\sigma(A)}(\bar{\chi})] &= \bigvee_{\nu\in\sigma(A)} \bigvee_{a,b_i\in A} \nu(a) \wedge \bigwedge_{i=1}^{n} \chi_i(b_i) \wedge [a \sim f^A(\bar{b})] \\
&= \bigvee_{\bar{b}\in A^n} \bigwedge_{i=1}^{n} \chi_i(b_i) \wedge \left(\bigvee_{a\in A} [a \sim f^A(\bar{b})] \wedge \left(\bigvee_{\nu\in\sigma(A)} \nu(a) \right) \right) \\
&= \bigvee_{\bar{b}\in A^n} \bigwedge_{i=1}^{n} \chi_i(b_i) \wedge \left(\bigvee_{a\in A} [a \approx a] \wedge [a \sim f^A(\bar{b})] \right) \\
&= \bigvee_{\bar{b}\in A^n} \bigwedge_{i=1}^{n} \chi_i(b_i) \wedge \bigwedge_{i=1}^{n} [b_i \approx b_i] \\
&= [\bar{\chi} \approx \bar{\chi}],
\end{aligned}
$$

where the third equality holds since for all $\nu \in \sigma(A)$ and $a \in A$, $\nu(a) \leq [a \approx a]$ and $\sigma_a \in \sigma(A)$, with $\sigma_a(a) = [a \approx a]$. The fourth equality follows by (M4) of morphism f^A and since $[a \sim f^A(\bar{b})] \leq [a \approx a]$. For a constant symbol c the proof is analog to the former, while for a relation symbol R it takes a simple calculation. Let's see that $\sigma(\alpha) : \sigma(A) \to \sigma(B)$ is a morphism of Ω-sets. (M1), (M2) and (M3) follow from definitions. For (M4) we note that if $\chi \in \sigma(A)$, then

$$
\begin{aligned}
\bigvee_{\chi\in\sigma(B)} [\chi \sim \sigma(\alpha)(\rho)] &= \bigvee_{\rho\in\sigma(B)} \bigvee_{a\in A, b\in B} \chi(b) \wedge \rho(a) \wedge [b \sim \alpha(a)] \\
&= \bigvee_{a\in A} \left(\rho(a) \wedge \left(\bigvee_{b\in B} [b \sim \alpha(a)] \wedge \left(\bigvee_{\rho\in\sigma(B)} \chi(b) \right) \right) \right) \\
&= \bigvee_{a\in A} \rho(a) \wedge \left(\bigvee_{b\in B} [b \sim \alpha(a)] \wedge [b \approx b] \right) \\
&= \bigvee_{a\in A} \rho(a) \wedge [a \approx a] \\
&= [\rho \approx \rho].
\end{aligned}
$$

If c is a constant symbol and R a relation symbol, the fact that morphism $\sigma(\alpha)$ is also a morphism of Ω-structures follows directly from definitions and the fact that α is morphism of Ω-structures. Let f be a symbol of function, we have that

$$
\begin{aligned}
&[\chi \sim f^{\sigma(B)} \circ (\sigma(\alpha), \dots, \sigma(\alpha))(\bar{\rho})] \\
&= \bigvee_{\bar{\chi} \in \sigma(B)^n} [\chi \sim f^{\sigma(B)}(\bar{\chi})] \wedge \bigwedge_{i=1}^{n} [\chi_i \sim \sigma(\alpha)(\rho_i)] \\
&= \bigvee_{\bar{\chi} \in \sigma(B)^n, b \in B, \bar{b}, \bar{c} \in B^n, \bar{a} \in A^n} \chi(b) \wedge [b \sim f^B(\bar{b})] \wedge \bigwedge_{i=1}^{n} \chi_i(c_i) \wedge \chi_i(b_i) \wedge \rho_i(a_i) \wedge [c_i \sim \alpha(a_i)] \\
&\leq \bigvee_{b \in B, \bar{b}, \bar{c} \in B^n, \bar{a} \in A^n} \chi(b) \wedge [b \sim f^B(\bar{b})] \wedge \bigwedge_{i=1}^{n} [b_i \approx c_i] \wedge [c_i \sim \alpha(a_i)] \wedge \rho_i(a_i) \\
&\leq \bigvee_{b \in B, \bar{b} \in B^n, \bar{a} \in A^n} \chi(b) \wedge [b \sim f^B(\bar{b})] \wedge [\bar{b} \sim (\alpha, \dots, \alpha)(\bar{a})] \wedge \bigwedge_{i=1}^{n} \rho_i(a_i) \\
&= \bigvee_{b \in B, \bar{a} \in A^n} \chi(b) \wedge [b \sim f^B \circ (\alpha, \dots, \alpha)(\bar{a})] \wedge \bigwedge_{i=1}^{n} \rho_i(a_i) \\
&= \bigvee_{b \in B, \bar{a} \in A^n} \chi(b) \wedge [b \sim \alpha \circ f^A(\bar{a})] \wedge \bigwedge_{i=1}^{n} \rho_i(a_i) \\
&= \bigvee_{b \in B, \bar{a} \in A^n} \chi(b) \wedge \bigwedge_{i=1}^{n} \rho_i(a_i) \wedge \bigvee_{a \in A} [b \sim \alpha(a)] \wedge [a \sim f^A(\bar{a})] \\
&= \bigvee_{b \in B, \bar{a} \in A^n, \nu \in \sigma(A), a, a' \in A} \chi(b) \wedge \nu(a) \wedge \nu(a') \wedge [b \sim \alpha(a)] \wedge [a \sim f^A(\bar{a})]) \wedge \bigwedge_{i=1}^{n} \rho_i(a_i) \\
&= \bigvee_{\nu \in \sigma(A)} [\chi \sim \sigma(\alpha)(\nu)] \wedge [\nu \sim f^{\sigma(A)}(\bar{\rho})] \\
&= [\chi \sim \sigma(\alpha) \circ f^{\sigma(A)}(\bar{\rho})],
\end{aligned}
$$

where the third last equalty holds since

$$
\bigvee_{a \in A} [b \sim \alpha(a)] \wedge [a \sim f^A(\bar{a})] = \bigvee_{\nu \in \sigma(A), a, a' \in A} \nu(a) \wedge \nu(a') \wedge [b \sim \alpha(a)] \wedge [a' \sim f^A(\bar{a})].
$$

The last equality follows from the definition of singletons. Finally, functoriality of σ follows by similar arguments. $\square$

The above theorem shows that every Ω-structure $\mathfrak{A}$ is isomorphic to $\sigma(\mathfrak{A})$, if we associate an Ω-structure $\mathfrak{A}$ to each presheaf, as we will do later, the closest sheaf to F will be the corresponding to $\sigma(\mathfrak{A})$. Although this result is not found in [FS79] is a direct generalization of the case for Ω-sets. We generalize the proof found in [Bo94, p. 157].

1.38 Theorem. *Let $\mathfrak{A}$ be an Ω-structure for τ. Then $\mathfrak{A}$ is isomorphic to the Ω-structure $\sigma(\mathfrak{A})$.*

Proof. We define $\alpha : A \to \sigma(A)$ and $\beta : \sigma(A) \to A$ such that, for all $\rho \in \sigma(A)$ and $a \in A$, we have $[\rho \sim \alpha(a)] = \rho(a)$ and $[a \sim \beta(\rho)] = \rho(a)$. In [Bo94, p. 157] it is shown that α and β are morphisms of Ω-sets, and that $\beta \circ \alpha = 1_A$ and $\alpha \circ \beta = 1_{\sigma(A)}$. Verifying that α, β define morphisms of Ω-structures require a simple but not short calculation. $\square$

The above result allows us to describe the relationship between the category of Ω-structures and its full subcategory of complete Ω-structures.

1.39 Corollary. *The category Ω-St_τ is equivalent to its full subcategory whose objects are complete Ω-structures.*

We can see each complete Ω-structure as a sheaf of structures on Ω by the next lemma which is ours. It is not proven in [FS79], since there is not a definition of sheaf of structures. The case $\tau = \emptyset$ follows easily from what is shown in [Bo94, p. 162-166]. Later on we prove that the defined functor is an isomorphism of categories by constructing an inverse.

1.40 Lemma. *We define a functor Σ between the category of complete Ω-structures and the category of sheaves of structures on Ω in the following way: let $\mathfrak{A}$ be a complete Ω-structure, we define a sheaf of structures for τ on Ω, $\Sigma(\mathfrak{A}) : \Omega \to St_\tau$.*

- *Let u be in Ω, define*
$$\Sigma(\mathfrak{A})(u) = \{a \in A \mid [a \approx a] = u\}.$$
- *Let c be a constant symbol, we define $c^{\Sigma(\mathfrak{A})(u)}$ as the unique element of A tsuch that, for all $a \in A$,*
$$[a \approx c^{\Sigma(\mathfrak{A})(u)}] = [a \sim c^{\mathfrak{A}}(*)] \wedge u.$$
- *Let f be a function symbol, we define $f^{\Sigma(\mathfrak{A})(u)}$ for all $\bar{a} \in \Sigma(\mathfrak{A})(u)^n$. We define $f^{\Sigma(A)(u)}(\bar{a})$ to be the unique element in A such that for all $b \in B$,*
$$[b \approx f^{\Sigma(\mathfrak{A})(u)}(\bar{a})] = [b \sim f^{\mathfrak{A}}(\bar{a})].$$
- *Let R be a relation symbol, for all $\bar{a} \in \Sigma(\mathfrak{A})(u)$ we define*
$$R^{\Sigma(\mathfrak{A})(u)}(\bar{a}) \textit{ iff } u = [R^{\mathfrak{A}}(\bar{a})].$$

- *Let $v \leq u$ be elements of Ω, we define the restrictions ρ^u_v for $m \in \Sigma(\mathfrak{A})(u)$, letting $m|_v$ to be the unique element of A such that, for all $b \in A$,*

$$[m|_v \approx b] = [m \approx b] \wedge v.$$

Let $\alpha : \mathfrak{A} \to \mathfrak{B}$ be a morphism of Ω-structures, where $\mathfrak{B}$ is a complete Ω-structure and $u \in \Omega$. We define $\Sigma(\alpha) : \Sigma(\mathfrak{A}) \to \Sigma(\mathfrak{B})$, a mapping such that for $a \in \Sigma(\mathfrak{A})(u)$, then $\Sigma(\alpha)_u(a)$ is the unique element of B such that, for all $b \in B$,

$$[b \approx \Sigma(\alpha)_u(a)] = [b \sim \alpha(a)].$$

Proof. Let u be in Ω, $v \leq u$, A, B be complete Ω-structures and $\alpha : A \to B$ be a morphism of Ω-structures, the fact that the definition of structure in $\Sigma(A)(u)$ and the definition of the mapping $\Sigma(\alpha)_u$ is correct follows from the fact that, for all$\bar{a} \in A^n$ and $d \in A$, the mappings

$$b \mapsto [b \sim c^A(*)] \wedge u, b \mapsto [b \sim f^A(\bar{a})], b \mapsto [b \sim \alpha(a)], b \mapsto [b \approx a] \wedge v$$

for $b \in A$, where c is a constant symbol and f function symbol all define singletons, hence, by using completeness we get that $c^{A(u)}, f^{A(u)}, \Sigma(\alpha)_u$ and ρ^u_v are well defined. The naturalness of the restrictions is proven in [Bo94, p. 164].

Next we show that the restrictions define morphisms of first order structures with signature τ. Let c be a constant symbol in τ, we have to see that$c^{\Sigma(A)(u)}|_v = c^{\Sigma(A)(v)}$, which follows from $u \wedge v = v$. Let f be a function symbol in τ and $\bar{a} \in \Sigma(u)^n$, we have to show that $f^{\Sigma(A)(u)}(\bar{a})|_v = f^{\Sigma(A)(v)}(a_1|_v, \ldots, a_n|_v)$. This follows since, for all $b \in A$,

$$[b \approx f^{\Sigma(A)(u)}(\bar{a})|_v] = [b \approx f^{\Sigma(A)(v)}(a_1|_v, \ldots, a_n|_v)],$$

which can be obtained by using $v = [a_i \approx a_i|_v]$. Let R be a relation symbol and $\bar{a} \in \Sigma(A)(u)^n$ such that $R^{\Sigma(A)(u)}(\bar{a})$, that is to say, $u = [R^A(\bar{a})]$. Using again $v = [a_i \approx a_i|_v]$ and that interpretations of relations are restricted and extensional we have $[R^A(a_1|_v, \ldots, a_n|_v)] = v$, that is $R^{\Sigma(A)(v)}(a_1|_v, \ldots, a_n|_v)$.

Next we show that $\Sigma(\alpha)_u$ is a homomorphism of structures. Let f be a function symbol in τ and $\bar{a} \in \Sigma(A)(u)^n$, we have

$$\begin{aligned}
[b \sim \alpha \circ f^A(\bar{a})] &= \bigvee_{d \in A} [b \sim \alpha(d)] \wedge [d \sim f^A(\bar{a})] \\
&= \bigvee_{d \in A} [b \sim \alpha(d)] \wedge [d \approx f^{\Sigma(A)(u)}(\bar{a})] \\
&= [b \sim \alpha(f^{\Sigma(A)(u)})(\bar{a})] \\
&= [b \approx \Sigma(\alpha)_u(f^{\Sigma(A)(u)}(\bar{a}))],
\end{aligned}$$

where the third last equality is obtained since the join is reached in $d = f^{\Sigma(A)(u)}(\bar{a})$. In a similar way $[b \sim f^b \circ (\alpha, \ldots, \alpha)(\bar{a})] = [b \approx f^{\Sigma(B)(u)}(\Sigma(\alpha)_u(a_1), \ldots, \Sigma(\alpha)_u(a_1))]$. By recalling that α is a morphism of Ω-structures and that B is a complete Ω-structure we have

$$\Sigma(\alpha)_u(f^{\Sigma(A)(u)}(\bar{a})) = f^{\Sigma(B)(u)}(\Sigma(\alpha)_u(a_1), \ldots, \Sigma(\alpha)_u(a_1))$$

as wanted. For constant symbols the calculation is very similar, and for relation symbols a routine computation is all that it is required.

It remains to show the functoriality of Σ. Let $\beta : B \to C$ be a morphism of (complete) Ω-structures, we note that $\Sigma(\beta \circ \alpha) = \Sigma(\beta) \circ \Sigma(\alpha)$, using the completeness of C. □

1.41 Lemma. *Let A be a complete Ω-set. The set*

$$A^n = \{(a_1, \ldots, a_n) \mid a_i \in A \textit{ and } [a_1 \approx a_1] = \cdots = [a_n \approx a_n]\}$$

with the Ω-equality given by

$$[\bar{a} \approx \bar{b}] = \bigwedge_{i=1}^{n} [a_i \approx b_i]$$

for all $\bar{a}, \bar{b} \in A^n$ and projections $p_i : A^n \to A$ given by

$$[a \sim p_i(\bar{a})] = [a \approx a_i]$$

for all $a \in A, \bar{a} \in A^n$ and $i \leq n$ is the product of n times A in the category of complete Ω-sets.

The following lemma, for $\tau = \emptyset$, that is to say that, for Ω-sets, appears in [Si01] as it has been said before. The general case proved below is ours.

1.42 Lemma. *We define a functor $\Gamma : St_\tau^{\Omega^{op}} \to \Omega\text{-}St_\tau$ in the following way: let F be a presheaf of structures on Ω. The set $\Gamma(F) = \coprod_{u \in \Omega} F(u)$ is an Ω-structure with the Ω-equality given by*

$$[a \approx b] = \bigvee \{w \in \Omega \mid a|_w = b|_w\},$$

para $a, b \in \Gamma(F)$; a constant symbol, c, is interpreted by making

$$[a \sim c^{\Gamma(F)}] = [a \approx c^{F(1)}] = \bigvee \{v \mid a|_v = c^{F(v)}\}$$

for all $a \in A$; a function symbol f are interpreted as the morphism $f^{\Gamma(F)} \mid \Gamma(F)^n \to \Gamma(F)$ defined by

$$\begin{aligned} [a \sim f^{\Gamma(F)}(a_1, \ldots, a_n)] &= [a \approx f^{F(\wedge_{i \leq n} u_i)}(a_1|_{\wedge_{i \leq n} u_i}, \ldots, a_n|_{\wedge_{i \leq n} u_i})] \\ &= \bigvee \left\{ w \in \Omega \mid a|_w = f^{F(w)}(a_1|_w, \ldots, a_n|_w) \right\} \end{aligned}$$

for all $a_i \in F(u_i)$ and $1 \leq i \leq n$; the interpretation of relation symbols R is given by the equation

$$[R^{\Gamma(F)}(\bar{a})] = \bigvee \left\{ w \leq \bigwedge_{i=1}^{n} [a_i \approx a_i] \mid R^{F(w)}(a_1|_w, \ldots, a_n|_w) \right\},$$

for all $\bar{a} \in A^n$. These interpretations turn $\Gamma(F)$ into an Ω-structure. Let G be another presheaf of structures for τ and $\alpha : F \to G$ morphism of presheaves, we define $\Gamma(\alpha) : \Gamma(F) \to \Gamma(G)$ by the equation

$$[b \sim \Gamma(\alpha)(a)] = [b \approx \alpha_u(a)] = \bigvee \{w \mid b|_w = \alpha_w(a|_w)\}$$

for all $b \in \Gamma(G)$ and $a \in F(u)$. Besides, if F is a sheaf of structures, $\Gamma(F)$ is a complete Ω-structure.

Proof. By a previous lemma, this Ω-equality defines an Ω-set that, in the case where F is a sheaf of sets, is complete. The fact that the above definition turns $\Gamma(F)$ into an Ω-structure reduces to similar calculations to those made before.

Let $\alpha : F \to G$ be a natural transformation between presheaves of structures. To see that $\Gamma(\alpha) : \Gamma(F) \to \Gamma(G)$ is a morphism of Ω-structures follows from the definitions. For example, if R is a relation symbol in τ, $\bar{a} \in \Gamma(F)^n$ and $\bar{b} \in \Gamma(G)^n$, then

$$\begin{aligned}
[R^{\Gamma(F)}(\bar{a})] \wedge [\bar{b} \sim (\alpha, \ldots, \alpha)(\bar{a})] &= [R^{\Gamma(F)}(\bar{a})] \wedge \bigwedge_{i=1}^{n} [b_i \sim \alpha(a_i)] \\
&= [R^{\Gamma(F)}(\bar{a})] \wedge \bigwedge_{i=1}^{n} \bigvee \{w \mid b_i|_{w_i} = \alpha_{w_i}(a_i|_{w_i})\} \\
&= \bigvee \{w \wedge \bigwedge_{i=1}^{n} w_i \mid R^{F(w)}(a_1|_w, \ldots, a_n|_w) \text{ y } b_i|_{w_i} = \alpha_{w_i}(a_i|_{w_i})\} \\
&\leq \bigvee \{u \mid R^{F(u)}(a_1|_u, \ldots, a_n|_u) \text{ y } b_i|_u = \alpha_u(a_i|_u)\} \\
&= \bigvee \{u \mid R^{G(u)}(b_1|_u, \ldots, b_n|_u)\} \\
&= [R^{\Gamma(G)}(\bar{b})].
\end{aligned}$$

Functoriality reduces to another simple calculation. □

1.43 Corollary. Γ *restricts to a functor between the category of sheaves of structures of signature τ on Ω and the category of complete Ω-structures of signature τ.*

To say that F is a sheave of structures is equivalent to say that for $\bar{a} \in \Gamma(F)^n$, $[R^{\Gamma(F)}(\bar{a})]$ is a maximum more than a join, that is to say

$$[R^{\Gamma(F)}\bar{a}] = \max \left\{ w \leq \bigwedge_{i=1}^{n} [a_i \approx a_i] \mid R^{F(w)}(a_1|_w, \ldots, a_n|_w) \right\}.$$

The following theorem, announced before, is proved as an equivalence in his version for Ω-sets in [Bo94, p. 162]. It also appears in [Si01, p. 2] and there is a proof of a minor variation, where the morphisms considered admit a simpler description.

1.44 Theorem. *The category of sheaves of structures of signature τ on Ω is isomorphic to the category of complete Ω-structures of signature τ.*

Proof. Since there is no place for confusion, let's call Γ the functor defined in the above corollary. We will prove that $\Gamma \circ \Sigma$ is the identity functor of the category of complete Ω-structures and that $\Sigma \circ \Gamma$ is the identity functor of the category of sheaves of structures over Ω. Let F be a sheaf of structures on a locale Ω and $u \in \Omega$, then

$$\Sigma \circ \Gamma(F)(u) = \{a \in \Gamma(F) \mid [a \approx a] = u\} = F(u).$$

It is easily proved that the restrictions defined by the functor $\Sigma \circ \Gamma$ are identical to those of F and that the structure of $\Sigma \circ \Gamma(F)(u)$ is in fact the same as the structure of $F(u)$. It is worth noting that in order to prove the facts corresponding to relation symbols it is necessary to use that F is a sheaf of structures. Finally, let G be a sheaf of structures and $\alpha : F \to G$, it is clear that $\Sigma \circ \Gamma(\alpha) = \alpha$.

Now we will see that $\Gamma \circ \Sigma$ is the identity of the category of complete Ω-structures. Let A be a complete Ω-structure, it is clear that $\Gamma \circ \Sigma(A) = A$. We have to show that the two Ω-equalities defined in A are the same. For now, let's denote by $[* \approx *]_1$ the Ω-equality of A and by $[* \approx *]_2$ the Ω-equality of $\Gamma \circ \Sigma(A)$. Let a, b be in A, then

$$[a \approx b]_2 = \bigvee \{w \leq [a \approx a]_1 \wedge [b \approx b]_1 \mid a|_w = b|_w\}.$$

For all $c \in A$, we have

$$[a|_{[a \approx b]_1} \approx c]_1 = [a \approx c]_1 \wedge [a \approx b]_1 = [b \approx c]_1 \wedge [a \approx b]_1 = [b|_{[a \approx b]_1} \approx c]_1.$$

On the other hand, if $w \in \Omega$ is such that $w \leq [a \approx a]_1 \wedge [b \approx b]_1$ and $a|_w = b|_w$ we have that

$$[a \approx a|_w]_1 = [a \approx a]_1 \wedge w = w = [a \approx b|_w]_1 = [a \approx b]_1 \wedge w$$

and then $w \leq [a \approx b]_1$. Hence $[a \approx b]_1 = [a \approx b]_2$ and we can forget about subindices.

By similar calculations to those carried out before, the Ω-structures A and $\Gamma \circ \Sigma(A)$ coincide. We can conclude also that, when B is a complete Ω-structure and $\beta : A \to B$, $\Gamma \circ \Sigma(\beta) = \beta$. $\square$

The following lemma is enunciated in [Si01, p. 39] and is easily proved.

1.45 Lemma. *Let $F \in St_\tau^{\Omega^{op}}$, $\rho \in \sigma \circ \Gamma(F)$ and $a \in \Gamma(F)$, then*

$$[\rho \approx \sigma_a] = \rho(a).$$

We have defined all the elements that play a roll in our definition of the sheafification functor whose description is found without proof, for the case of Ω-sets, in [Si01, p. 48], it is not found in [FS79]. The proof is ours.

1.46 Theorem. *The inclusion i of the category of sheaves of structures of signature τ on Ω into $St_\tau^{\Omega^{op}}$ admits $\Sigma \circ \sigma \circ \Gamma$ as a left adjoint.*

Proof. Let F be in $St_\tau^{\Omega^{op}}$, we define a morphism of presheaves $\eta^F : F \to i \circ \Sigma \circ \sigma \circ \Gamma(F)$ (from now on we will omit the morphism i). Let u be an element of Ω, then

$$\begin{aligned} \eta_u^F : F(u) &\to \Sigma \circ \sigma \circ \Gamma(u) \\ a &\mapsto \sigma_a. \end{aligned}$$

Clearly this mapping is well defined since, for all $a \in F(u)$, $[\sigma_a \approx \sigma_a] = \sigma_a(a) = [a \approx a]$. It is also easy to see that this is in fact a morphism of structures by using that $\sigma \circ \Gamma(F)$ is complete and doing some calculations similar to those made before. A routine computation shows that η^F is a natural transformation.

Now we show that η is the unity in the adjunction. Let H be a sheaf of structures of signature τ on Ω and $\beta : F \to H$ a morphism of presheaves. Let's see that there is a unique $\gamma : \Sigma \circ \sigma \circ \Gamma(F) \to H$ such that $\gamma \circ \eta^F = \beta$. Let u be a fixed element of Ω. Let ρ be an element of $\Sigma \circ \sigma \circ \Gamma(F)(u)$. We know that $[\rho \approx \rho] = \bigvee_{a \in \Gamma(F)} \rho(a) = u$. We will define $\gamma_u(\rho)$, by using that H is a sheaf, as the gluing of the compatible family $(\beta_{\rho(a)}(a|_{\rho(a)}))_{a \in \Gamma(F)}$. This family makes sense since for all $a \in \Gamma(F)$, $\rho(a) \leq [a \approx a]$. Let's see that the family is compatible. Let a, b be in $\Gamma(F)$, then

$$\begin{aligned} \beta_{\rho(a)}(a|_{\rho(a)})|_{\rho(a)\wedge\rho(b)} &= \beta_{\rho(a)\wedge\rho(b)}(a|_{\rho(a)\wedge\rho(b)}) \\ &= \beta_{\rho(a)\wedge\rho(b)}(b|_{\rho(a)\wedge\rho(b)}) \\ &= \beta_{\rho(b)}(b|_{\rho(b)})|_{\rho(a)\wedge\rho(b)}, \end{aligned}$$

where we have used that β is a natural transformation and H is a sheaf. We note that since F is not necessarily a separated sheaf it could be the case that $a|_{[a\approx b]} \neq b|_{[a\approx b]}$, however since H is a sheaf the above equalities hold. We can define $\gamma_u(\rho)$ as the unique element in $H(u)$ such that, for all $a \in \Gamma(F)$

$$\gamma_u(\rho)|_{\rho(a)} = \beta_{\rho(a)}(a|_{\rho(a)}).$$

We have that $\gamma \circ \eta^F = \beta$ since, for all $a \in F(u)$, we have that for all $b \in \Gamma(F)$

$$\beta_u(a)|_{[a\approx b]} = \beta_{[a\approx b]}(a|_{[a\approx b]}) = \beta_{[a\approx b]}(b|_{[a\approx b]}),$$

and so $\gamma_u(\sigma_a) = \beta_u(a)$, that is to say, $\gamma_u \eta^F_u(a) = \beta_u(a)$.

Next we see that γ_u is a homomorphism of structures. The fact that it preserves interpretations of constant symbols follows in a similar say to the preservation of function and relation symbols. Let f be a function symbol of τ and $\bar{\rho} \in \Sigma \circ \sigma \circ \Gamma(F)(u)^n$. We have to show

$$\gamma_u(f^{\Sigma\circ\sigma\circ\Gamma(F)(u)}(\bar{\rho})) = f^{H(u)}(\gamma_u(\rho_1), \ldots, \gamma_u(\rho_n)).$$

Let's denote $f^{\Sigma\circ\sigma\circ\Gamma(F)(u)}(\bar{\rho}) = \rho$ and let a be an element of $\Gamma(F)$, we have that

$$\begin{aligned} \rho(a) &= [\sigma_a \approx f^{\Sigma\circ\sigma\circ\Gamma(F)(u)}(\bar{\rho})] \\ &= \bigvee_{b\in\Gamma(F), \bar{a}\in\Gamma(F)^n} [b \sim f^{\Gamma(F)}(\bar{a})] \wedge [b \approx a] \wedge \bigwedge_{i=1}^{n} \rho_i(a_i). \end{aligned}$$

Let now b be in $\Gamma(F)$, $\bar{a} \in \Gamma(F)^n$ and $w(b, \bar{a}) = [b \sim f^{\Gamma(F)}(\bar{a})] \wedge [b \approx a] \wedge \bigwedge_{i=1}^n \rho_i(a_i)$. We have that

$$\begin{aligned} f^{H(u)}(\gamma_u(\rho_1), \ldots, \gamma_u(\rho_n))|_{w(b,\bar{a})} &= f^{H(w(b,\bar{a}))}(\gamma_u(\rho_1)|_{w(b,\bar{a})}, \ldots, \gamma_u(\rho_n)|_{w(b,\bar{a})}) \\ &= f^{H(w(b,\bar{a}))}(\beta_{w(b,\bar{a})}(a_1|_w), \ldots, \beta_{w(b,\bar{a})}(a_n|_w)) \\ &= \beta_{w(b,\bar{a})}(f^{F(w(b,\bar{a}))}(a_1|_w, \ldots, a_n|_w)) \\ &= \beta_{w(b,\bar{a})}(b|_{w(b,\bar{a})}) \\ &= \beta_{w(b,\bar{a})}(a|_{w(b,\bar{a})}) \\ &= \beta_{\rho(a)}(a|_{\rho(a)})|_{w(b,\bar{a})}, \end{aligned}$$

where we have use the definition of $w(b, \bar{a})$.

Since $(w(b, \bar{a}))_{b\in\Gamma(F),\bar{a}\in\Gamma(F)^n}$ constitutes a covering of $\rho(a)$ we have that

$$f^{H(u)}(\gamma_u(\rho_1), \ldots, \gamma_u(\rho_n))|_{\rho(a)} = \beta_{\rho(a)}(a|_{\rho(a)})$$

and so

$$f^{H(u)}(\gamma_u(\rho_1), \ldots, \gamma_u(\rho_n)) = \gamma_u(f^{H(u)}(\gamma_u(\rho_1), \ldots, \gamma_u(\rho_n)))$$

as we wanted to show.

Let R be a relation symbol of τ and $\bar{\rho} \in \Sigma \circ \sigma \circ \Gamma(F)(u)^n$ such that $R^{\Sigma\circ\sigma\circ\Gamma(F)(u)}(\bar{\rho})$, we have that

$$\begin{aligned} u &= [R^{\sigma\circ\Gamma(F)}(\bar{\rho})] \\ &= \bigvee_{\bar{a}\in\Gamma(F)^n} [R^{\Gamma(F)}(\bar{a})] \wedge \bigwedge_{i=1}^{n} \rho_i(a_i) \\ &= \bigvee_{\bar{a}\in\Gamma(F)^n} \bigvee\{w \wedge \bigwedge_{i=1}^{n} \rho_i(a_i) : R^{F(w)}(b_1|_w, \ldots, b_n|_w)\} \end{aligned}$$

Let $\bar{a} \in \Gamma(F)^n$ and w such that $R^{F(w)}(a_1|_w, \ldots, a_n|_w)$, then

$$R^{F(w\wedge\bigwedge_{i\leq n}\rho_i(a_i))}(a_1|_{w\wedge\bigwedge_{i\leq n}\rho_i(a_i)}, \ldots, a_n|_{w\wedge\bigwedge_{i\leq n}\rho_i(a_i)}),$$

and so, by using that β is a morphism of presheaves of structures we obtain

$$R^{H(w\wedge\bigwedge_{i\leq n}\rho_i(a_i))}(\beta_{w\wedge\bigwedge_{i\leq n}\rho_i(a_i)}(a_1|_{w\wedge\bigwedge_{i\leq n}\rho_i(a_i)}), \ldots, \beta_{w\wedge\bigwedge_{i\leq n}\rho_i(a_i)}(a_n|_{w\wedge\bigwedge_{i\leq n}\rho_i(a_i)}));$$

by using the definition of γ we get

$$R^{H(w\wedge\bigwedge_{i\leq n}\rho_i(a_i))}(\gamma_u(\rho_1)|_{w\wedge\bigwedge_{i\leq n}\rho_i(a_i)}, \ldots, \gamma_u(\rho_n)|_{w\wedge\bigwedge_{i\leq n}\rho_i(a_i)}).$$

And then, since the elements $w \wedge \bigwedge_{i\leq n}\rho_i(a_i)$ constitute a covering of u, and H is a sheaf, we can conclude that

$$R^{H(u)}(\gamma_u(\rho_1), \ldots, \gamma_u(\rho_n)).$$

Let's now see that γ is a natural transformation. Let $v \leq u$, and $\rho \in \Sigma \circ \sigma \circ \Gamma(F)$ we show $\gamma_u(\rho)|_v = \gamma_v(\rho|_v)$. A covering for v is $(\rho(a) \wedge v)_{a\in A}$, besides

$$\begin{aligned} \rho|_v(a|_v) &= [\rho|_v \approx \sigma_{a|_v}] \\ &= [\rho \approx \sigma_{a|_v}] \wedge v \\ &= \rho(a|_v) \wedge v \\ &= \rho(a|_v), \end{aligned}$$

and so

$$\begin{aligned} \gamma_v(\rho|_v)|_{\rho(a|_v)} &= \gamma_v(\rho|_v)|_{\rho|_v(a|_v)} \\ &= \beta_{\rho|_v(a|_v)}((a|_v)|_{\rho|_v(a|_v)}) \\ &= \beta_{\rho(a|_v)}(a|_{\rho(a|_v)}) \\ &= \gamma_u(\rho)|_{\rho(a|_v)}, \end{aligned}$$

since H is a sheaf, we conclude $\gamma_u(\rho)|_v = \gamma_v(\rho|_v)$.

It remains to prove that γ is the only transformation $\gamma : \Sigma \circ \sigma \circ \Gamma(F) \to H$ such that $\gamma \circ \eta^F = \beta$. Let $\alpha : \Sigma \circ \sigma \circ \Gamma(F) \to H$ be a morphism of presheaves of structures such that $\alpha \circ \eta^F = \beta$, that is, for all $v \in \Omega$ and $b \in F(v)$

$$\alpha_v(\sigma_b) = \beta_v(b).$$

Let a be an element of $\Gamma(F)$ and $\rho \in \Sigma \circ \sigma \circ \Gamma(F)(u)$. We have that

$$\alpha_u(\rho)|_{\rho(a)} = \beta_{\rho(a)}(a|_{\rho(a)}),$$

$$\begin{aligned}
\alpha_u(\rho)|_{\rho(a)} &= \alpha_{\rho(a)}(\rho|_{\rho(a)}) \\
&= \alpha_{\rho(a)}(\rho|_{[\rho \approx \sigma_a]}) \\
&= \alpha_{\rho(a)}((\sigma_a)|_{\rho(a)}) \\
&= \alpha_{\rho(a)}(\sigma_{a|_{\rho(a)}}) \\
&= \beta_{\rho(a)}(a|_{\rho(a)}),
\end{aligned}$$

and so $\alpha = \gamma$. This concludes the proof of the uniqueness of γ. □

3.2 Logic of sheaves of structures

In [FS79, p. 358] it is defined a first order semantics for Ω-structures. However, since our main interest is to generalize Caicedo's work, we will define a semantic for sheaves of structures, equivalently complete Ω-structures. From now on we consider sheaves of structures $\mathfrak{A}$, unless is stated. We will treat them as a functor letting $A(u)$ denote the structure corresponding to each $u \in \Omega$ and as a complete Ω-structure with universe $A = \coprod_{u \in \Omega} A(u)$.

2.1 Definition (Interpretation of terms.)**.** Let $t(x_1, \ldots, x_n)$ be a term with n variables in the language τ and $\mathfrak{A}$ an Ω-structure. We define by recursion in the complexity of the term a function $t^{\mathfrak{A}} : A^n \to A$ that we call the interpretation of t in $\mathfrak{A}$.

- If c is a constant symbol $c^{\mathfrak{A}} = c^{A(1)}$.
- If x is a variable then $x^{\mathfrak{A}} : A \to A$ is the identity.
- If $t_1^{\mathfrak{A}}, \ldots, t_n^{\mathfrak{A}}$ have been defined and f is a function symbol then, for all $\bar{a} \in A^{n,}$ con $a_i \in u_i$ $f(t_1, \ldots, t_n)^{\mathfrak{A}}(\bar{a}) = f^{A(\bigwedge_{i \leq n} u_i)} \circ (t_1^A(\bar{a}), \ldots, t_n^A(\bar{a}))$.

If t has n variables and appears in a formula with more free variables, we will consider the rest of the variables as mute for the interpretation.

Instead of defining a logic beginning with points, as in [Ca95], we define the maximum of the elements in the algebra which satisfy a formula.

2.2 Definition (First order semantics for sheaves of structures.)**.** Let $\mathfrak{A}$ be a sheaf of structures. We define a semantics valuated in the complete Heyting Algebra Ω. For every formula ϕ with n free variables we define

$$\phi^{\mathfrak{A}} : A^n \to \Omega, \bar{a} \mapsto [\phi^{\mathfrak{A}}(\bar{a})]$$

by recursion in the complexity of the formula in the following way:

- Let t_1, t_2 be terms with n and m free variables and $\bar{a} \in A^n, \bar{b} \in A^m$,

$$[(t_1(\bar{a}) = t_2(\bar{b}))^{\mathfrak{A}}] = [(t_1^{\mathfrak{A}}(\bar{a}) \approx t_2^{A}(\bar{b}))].$$

- Let R be an n-ary symbol of relation in τ and $t_1, \ldots, t_n$ terms and $\bar{a} \in A^m$ where m is the number of variables of $R(t_1, \ldots, t_n)$,

$$[R(t_1(\bar{a}), \ldots, t_n(\bar{a}))^{\mathfrak{A}}] = [R^A(t_1^{\mathfrak{A}}(\bar{a}), \ldots, t_n^{\mathfrak{A}}(\bar{a}))].$$

- If ϕ, ψ are formulas, we define

  $[(\neg\phi)^{\mathfrak{A}}] = \neg[\phi^{\mathfrak{A}}]$,

  $[(\phi \to \psi)^{\mathfrak{A}}] = [\phi^{\mathfrak{A}}] \Rightarrow [\psi^{\mathfrak{A}}]$,

  $[(\phi \wedge \psi)^{\mathfrak{A}}] = [\phi^{\mathfrak{A}}] \wedge [\psi^{\mathfrak{A}}]$,

  $[(\phi \vee \psi)^{\mathfrak{A}}] = [\phi^{\mathfrak{A}}] \vee [\psi^{\mathfrak{A}}]$,

  $[(\exists x \phi(x))^{\mathfrak{A}}] = \bigvee_{a \in A}[\phi^{\mathfrak{A}}(a)] \wedge [a \approx a]$,

  $[(\forall x \phi(x))^{\mathfrak{A}}] = \bigwedge_{a \in A}[a \approx a] \Rightarrow [\phi^{\mathfrak{A}}(a)]$.

The interpretations of formulas in complete Ω-structures turn out to be extensional in the same way that interpretations of relations are. If ϕ is a formula in the language τ, $\mathfrak{A}$ is a sheaf of structures of signature τ and $a_i, b_i \in A$, then

$$[\phi^A(\bar{a})] \wedge \bigwedge_{i=1}^{n}[a_i \approx b_i] \leq [\phi^A(\bar{b})].$$

By an easy calculation we can obtain the satisfaction of the axioms of equality for first order theories in all sheaves of structures.

2.3 Lemma. *Let $\phi(x_1, \ldots, x_n)$ be a formula with n free variables and $\mathfrak{A}$ a complete Ω-structure with $\bar{a}, \bar{b} \in A^n$, then*

- $[\forall x(x = x)^{\mathfrak{A}}] = 1$
- $[\forall x_1 \ldots \forall x_n \forall y_1 \ldots \forall y_n((\phi(\bar{x}) \wedge \bigwedge_{i=1}^{n} x_i = y_i) \rightarrow \phi(\bar{y}))^{\mathfrak{A}}] = 1$

Following an inverse order to the one used in [Ca95] we define the forcing in elements of the locale.

2.4 Definition (Forcing in elements of Ω)**.** Let $\mathfrak{A}$ be a complete Ω-structure, $a_i \in A$ and $\phi(\bar{x})$ be a formula with n fre variables. We define the forcing in elements of Ω by

$$\mathfrak{A} \Vdash_u \phi(\bar{a}) \text{ iff } u \leq [\phi^{\mathfrak{A}}(a_1, \ldots, a_n)].$$

From now on we generalize the logic defined in [Ca95] following its lines until the proof of the Generic Model Theorem. We begin by noting that:

$$[\phi(\bar{a})^{\mathfrak{A}}] = \max\{u \in \Omega \mid \mathfrak{A} \Vdash_u \phi(\bar{a})\}.$$

We have, as in [Ca95, p. 18]:

(A) If $\mathfrak{A} \Vdash_u \phi$ and $v \leq u$, then $\mathfrak{A} \Vdash_v \phi$.

(B) If $a_j \in A(u)$ and $u = \bigvee_{i \in I} u_i$ and $\mathfrak{A} \Vdash_{u_i} \phi(a_1|_{u_i}, \ldots, a_n|_{u_i})$ for all $i \in I$, then $\mathfrak{A} \Vdash_{\bigvee_{i \in I} u_i} \phi(\bar{a})$.

We note that (B) generalizes for all formulas of language τ what holds by definition for atomic formulas. Our forcing generalizes the one defined in [Ca95, p. 19] as it is shown in the following lemma.

2.5 Lemma (Kripke-Joyal semantics)**.** *Let $\mathfrak{A}$ be a complete Ω-structure. The forcing relation in the elements u of the algebra Ω, $\mathfrak{A} \Vdash_u \phi(\bar{a})$, is the only one which satisfies the following conditions.*

1. $\mathfrak{A} \Vdash_u t_1(\bar{a}) = t_2(\bar{b})$ *iff* $t_1^{\mathfrak{A}}(a_1|_u, \ldots, a_n|_u) = t_2^{\mathfrak{A}}(b_1|_u, \ldots, b_n|_u)$.
2. $\mathfrak{A} \Vdash_u R(t_1(\overline{a_1}), \ldots, t_n(\overline{a_n}))$ *iff* $R^{A(u)}(t_1^{\mathfrak{A}}(\overline{a_1})|_u, \ldots, t_n^{\mathfrak{A}}(\overline{a_n})|_u)$.
3. $\mathfrak{A} \Vdash_u \phi \wedge \psi$ *iff* $\mathfrak{A} \Vdash_u \phi$ *y* $\mathfrak{A} \Vdash_u \psi$.
4. $\mathfrak{A} \Vdash_u \phi \vee \psi$ *iff there are* $v, w \in \Omega$ *such that* $u = v \vee w$ *with* $\mathfrak{A} \Vdash_u \phi$ *and* $\mathfrak{A} \Vdash_w \psi$.

5. $\mathfrak{A} \Vdash_u \neg\phi$ *iff for all* $w \in \Omega$ *such that* $0 < w \leq u$, $\mathfrak{A} \nVdash_w \phi$.

6. $\mathfrak{A} \Vdash_u \phi \to \psi$ *iff for all* $w \leq u$, $\mathfrak{A} \Vdash_w \phi$ *implies* $\mathfrak{A} \Vdash_w \psi$.

7. $\mathfrak{A} \Vdash_u \exists x(\phi(x, \bar{a}))$ *there is* $\{u_i\}_{i\in I}$ *in* Ω *and elements* $b_i \in A(u_i)$ *such that* $u = \bigvee_{i\in I} u_i$ *and for all* $i \in I$, $\mathfrak{A} \Vdash_{u_i} \phi(b_i, \bar{a}))$.

8. $\mathfrak{A} \Vdash_u \forall x(\phi(x, \bar{a}))$ *iff for all* $w \leq u$ *and all* $b \in A(w)$, $\mathfrak{A} \Vdash_w \phi(b, \bar{a})$.

Proof. The uniqueness in the relation of forcing $\mathfrak{A} \Vdash_u \phi$ will be proved if 1.-8. are satisfied since these constitute a definition by recursion in the complexity of a formula ϕ of the forcing relation. 1. and 2. follow directly from extensionality and restriction of the interpretation of R and given that the interpretation of terms is a morphism of complete Ω-structures. The remaining cases are proved by induction in the complexity of ϕ.

3. $u \leq [(\phi \wedge \psi)^{\mathfrak{A}}] = [\phi^{\mathfrak{A}}] \wedge [\psi^{\mathfrak{A}}]$ iff $u \leq [\phi^{\mathfrak{A}}]$ and $u \leq [\psi^{\mathfrak{A}}]$.

4. $u \leq [(\phi \vee \psi)^{\mathfrak{A}}] = [\phi^{\mathfrak{A}}] \vee [\psi^{\mathfrak{A}}]$ iff $u = u \wedge ([\phi^{\mathfrak{A}}] \vee [\psi^{\mathfrak{A}}])$, this is equivalent to the existence of v, w such that $u = w \vee v$, with $w \leq [\phi^{\mathfrak{A}}]$ and $v \leq [\psi^{\mathfrak{A}}]$. In that case $w \leq u \wedge [\phi^{\mathfrak{A}}]$ and $v \leq [\psi^{\mathfrak{A}}]$, and hence $u \leq w \wedge v \leq u \wedge [\phi^{\mathfrak{A}}] \vee \wedge[\psi^{\mathfrak{A}}] \leq u$.

5. $u \leq [(\neg\phi)^{\mathfrak{A}}] = \neg[\phi^{\mathfrak{A}}] = \bigvee\{b : b \wedge [\phi^{\mathfrak{A}}] = 0\}$ is equivalent to $u \wedge [\phi^{\mathfrak{A}}] = 0$, which is equivalent to say that for all w such that $0 < w \leq u$, $w \wedge [\phi^{\mathfrak{A}}] = 0$. This happens iff for all $0 < w \leq u$ we do not have $w \leq [\phi^{\mathfrak{A}}]$. The last equivalence holds since, given $0 < w \leq u$, if $w \leq [\phi^{\mathfrak{A}}]$, then $w \wedge [\phi^{\mathfrak{A}}] = w > 0$.

6. $u \leq [(\phi \to \psi)^{\mathfrak{A}}]$ iff $u \wedge [\phi^{\mathfrak{A}}] \leq [\psi^{\mathfrak{A}}]$, that is to say, for all $w \leq u$, if $w \leq [\phi^{\mathfrak{A}}]$, then $w \leq [\psi^{\mathfrak{A}}]$.

7. If $u \leq [(\exists x(\phi(x, \bar{a})))^{\mathfrak{A}}] = \bigvee_{b\in A}[\phi^{\mathfrak{A}}(b, \bar{a})] \wedge [b \approx b]$, we have that $u = \bigvee_{b\in A} u \wedge [\phi^{\mathfrak{A}}(b, \bar{a})] \wedge [b \approx b]$, if we define $u_b = [\phi^{\mathfrak{A}}(b, \bar{a})] \wedge u \wedge [b \approx b]$ we have, for all $b \in A$, $b|_{u_b} \in A(u_b)$, $u_b \leq [\phi^{\mathfrak{A}}(b|_{u_b}, \bar{a})]$, and $u = \bigvee_{b\in A} u_b$. On the other hand, if $u = \bigvee_{i\in I} u_i$ and there are $b_i \in A(u_i)$ such that $u = \bigvee_{i\in I} u_i$ and for all $i \in I$ $u_i \leq [\phi^{\mathfrak{A}}(b_i, \bar{a})]$, then, since $[b_i \approx b_i] = u_i$, $u \leq \bigvee_{i\in I}[\phi^{\mathfrak{A}}(b_i, \bar{a})] \wedge [b_i \approx b_i] \leq [(\exists x(\phi(x, \bar{a})))^{\mathfrak{A}}]$.

8. Suppose $u \leq [(\forall x(\phi(x, \bar{a})))^{\mathfrak{A}}] = \bigwedge_{b\in A}[b \approx b] \Rightarrow [\phi(b, \bar{a})^{\mathfrak{A}}]$. Let w be an element of Ω such that $w \leq u$ and $b \in A(w)$, then $u \leq [b \approx b] \Rightarrow [\phi(b, \bar{a})^{\mathfrak{A}}]$ so that $w = w \wedge u \leq [\phi(b, \bar{a})^{\mathfrak{A}}]$. Now assume that for all $w \leq u$ and all $b \in A(w)$, $w \leq [\phi^{\mathfrak{A}}(b, \bar{a})]$. Let c be in A, we have that $w = [c \approx c] \wedge u \leq u$, so $w \leq [\phi(b|_w, a_1|_w, \ldots, a_n|_w)^{\mathfrak{A}}]$, that is to say $u \wedge [b \approx b] \leq [\phi(b, \bar{a})^{\mathfrak{A}}]$, which implies $u \leq [b \approx b] \Rightarrow [\phi^{\mathfrak{A}}(b, \bar{a})]$. Hence $u \leq [(\forall x(\phi(x, \bar{a})))^{\mathfrak{A}}]$. □

2.6 Example. Let R be a commutative ring with unity and $\mathfrak{A}$ the sheaf of section of its structural space that, as it has been said, is a sheaf of rings. The fact that it is a sheaf of rings implies that, in the Kripke-Joyal semantics $\mathfrak{A}$ satisfies all the first order axioms for the theory of rings, furthermore, $\mathfrak{A}$ satisfies the first order sentence that says that it is a local ring, that is:

$$\mathfrak{A} \Vdash_1 \forall x(\exists y(xy = 1) \vee \exists y((1-x)y = 1)).$$

In order to see this let u be an element of $Sp(R)$, we have to prove that for every $s \in A(u)$

$$\mathfrak{A} \Vdash_u \exists y((s.y = 1) \vee \exists y((1-s)y = 1))$$

Since u is an union of basic open sets and by (B), it remains to consider the case where $u = \mathcal{O}_b$ for some $b \in R$. In this case $s = s^a_{b^n}$ for some $a \in R$ [Bo94, p. 181]. Now, given a prime ideal $J \in \mathcal{O}_b$, if $a \in J$, then $b^n - a \notin J$, because if not, then $b^n = a + (b^n - a) \in J$, which is impossible since J is a radical ideal. In this way we have $\mathcal{O}_b = (\mathcal{O}_b \cap \mathcal{O}_a) \cup (\mathcal{O}_b \cap \mathcal{O}_{b^n - a})$, besides, it is evident from the definitions that

$$\mathfrak{A} \Vdash_{\mathcal{O}_b \cap \mathcal{O}_a} s^a_{b^n} s^{b^n}_a = 1, \mathfrak{A} \Vdash_{\mathcal{O}_b \cap \mathcal{O}_{b^n-a}} (1 - s^a_{b^n}) s^{b^n}_{b^n-a} = 1,$$

where the above assertions are evident if we recall that the interpretation of 1 in $A(O_{b^n-a})$ is $s^{b^n-a}_{b^n-a}$, and so the last satisfaction reduces to a simple calculation. We obtain what we want by applying the laws of the above lemma to the logic connectives $\exists$ and $\vee$.

In this way, by studying the semantics of the structural space or the ring R, we realize that this space behaves as a local ring, although R might not be local. On the other hand, we know that the ring of global sections $A(Sp(R))$ is isomorphic to the ring R [Bo94, p. 182]. Another way to look at this phenomenon is in the form of a local statement, such as the theorem of existence of geodesics for surfaces or Picard theorem, they both guarantee certain existence in the neighborhood of a point, that is to say, locally. We could then say that locally, any ring is local. The price to pay for winning this good behavior is that the logic of sheaves of structures does not satisfy all the laws of classical logic, as we will see later on.

After establishing this equivalence we will prove that Caicedo's results still hold for sheaves on locales by adapting his proofs to our case. In what follows, $\mathfrak{A}$ will denote a complete Ω-structure unless stated. The following lemma in its version for topological sheaves is found in [Ca95, p. 17].

2.7 Lemma (Double negation.). *Let be $u \in \Omega$. $\mathfrak{A} \Vdash_u \neg\neg\phi(\bar{a})$ is equivalent to $\downarrow (u \wedge [\phi(\bar{a})])$ being dense in $\downarrow u$.*

Proof. Let's assume that $u \leq \neg\neg[\phi(\bar{a})^{\mathfrak{A}}]$, this is equivalent to $u \wedge \neg[\phi(\bar{a})] = u \wedge \bigvee\{w \mid w \wedge [\phi(\bar{a})^{\mathfrak{A}} = 0]\} = 0$, it is easy to see that

$$\begin{aligned} \neg(u \wedge [\phi(\bar{a})^{\mathfrak{A}}]) &= \bigvee\{b \mid b \wedge [\phi(\bar{a})^{\mathfrak{A}}] = 0\} \\ &= \bigvee\{b \mid b \wedge u = 0\}, \end{aligned}$$

that is, $\neg(u \wedge [\phi(\bar{a})]) = \neg u$, so $\neg(u \wedge [\phi(\bar{a})^{\mathfrak{A}}]) \wedge u = \neg u \wedge u = 0$, and then $\downarrow (u \wedge [\phi(\bar{a})])$ is dense in $\downarrow u$.

Let us assume now that $u \wedge \neg(u \wedge [\phi(\bar{a})]^{\mathfrak{A}}) = 0$, we have $u \wedge [\phi(\bar{a})] \leq [\phi(\bar{a})^{\mathfrak{A}}]$, this implies $\neg[\phi(\bar{a})^{\mathfrak{A}}] \leq \neg(u \wedge [\phi(\bar{a})^{\mathfrak{A}}])$, so $u \wedge \neg[\phi(\bar{a})^{\mathfrak{A}}] = 0$, and we conclude $u \leq \neg\neg[\phi(\bar{a})^{\mathfrak{A}}]$. ☐

2.8 Example. [Ca95, p. 16] Let's consider the topological space given by $X = \mathbb{R} \cup \{a\}$, con $a \notin \mathbb{R}$, where we add to the topology of $\mathbb{R}$ the neighborhoods of a given by $(U \setminus \{0\}) \cup \{a\}$ for each open U which contains 0. We consider the topological sheaf given by the tale space $p : X \to \mathbb{R}$, where, for all $x \in \mathbb{R}$, $p(x) = x$ and $p(a) = 0$. Let's consider the global sections $s : \mathbb{R} \to X$ such that $s(x) = x$ and $r : \mathbb{R} \to X$ such that $r(x) = x$ para $x \neq 0$ y $s(0) = a$. In the logic of the associated sheaf of structures $\mathfrak{A}$ we have that $\mathfrak{A} \nVdash_{\mathbb{R}} s = r$, but $\mathfrak{A} \Vdash_{\mathbb{R}} \neg\neg s = r$ since $(-\infty, 0) \cup (0, \infty)$ is dense in $\mathbb{R}$. Besides, since $[(s = r)^{A}] = (-\infty, 0) \cup (0, \infty)$ we have $[\neg(s = r)^{A}] = \emptyset$, in particular $\mathfrak{A} \nVdash_{\mathbb{R}} (s = r) \vee \neg(s = r)$.

2.9 Corollary. $\mathfrak{A} \Vdash_u \neg\neg\phi(\bar{a})$ *iff there is* $w \leq u$ *such that* $\downarrow w$ *is dense in* $\downarrow u$ *and* $\mathfrak{A} \Vdash_w \phi(\bar{a})$.

The following result is analogous to the one proved in [Ca95, p. 18]. In its proof we use the axiom of choice.

2.10 Theorem (Maximum principle)**.** *If* $\mathfrak{A} \Vdash_u \exists x\phi(x)$, *then there is* $w \leq u$, *with* $\downarrow w$ *is dense in* $\downarrow u$ *and* $a \in A(w)$ *such that* $\mathfrak{A} \Vdash_w \phi(a)$.

Proof. Assume that $\mathfrak{A} \Vdash_u \exists x\phi(x)$, then there is a covering $u = \bigvee_{i \in I} u_i$ and elements $a_i \in A(u_i)$ such that $u_i \leq [\phi(a_i)^{\mathfrak{A}}]$ for all $i \in I$. Let's consider the set

$$\Gamma = \{a \mid [a \approx a] \leq u \wedge [\phi(a)^{\mathfrak{A}}]\} \neq \emptyset.$$

We define the partial order $\subseteq$ for all $a, b \in \Gamma$ in the following way

$$a \subseteq b \text{ iff } [a \approx a] \leq [b \approx b] \text{ and } b|_{[a \approx a]} = a.$$

Suppose that $\{a_j\}_{j\in J}$, with $J \subseteq I$, is a chain for the order $\subseteq$. This implies that it is a compatible family. Hence there is a unique gluing $a \in \bigvee_{j\in J} u_j$ such that $a|_{u_j} = a_j$, and so $a_j \subseteq a$. Furthermore, we have $\mathfrak{A} \Vdash_{u_j} \phi(a|_{u_j})$, and so $\mathfrak{A} \Vdash_{\bigvee u_j} \phi(a)$. We conclude that all chain has an upper bound, by Zorn's lemma there is an element a which is maximal in Γ, with $[a \approx a] = v \leq u$.

Let's now see that $\downarrow v$ is dense in $\downarrow u$, or what is the same $\neg v \wedge u = 0$. In other case $\neg v \wedge \bigvee_{i\in I} u_i > 0$, and then there is $i \in I$ such that $\neg v \wedge u_i > 0$. We would have then $v < (v \vee \neg v \wedge u_i)$, or else

$$0 = v \wedge \neg v = \neg v \wedge (v \vee (\neg v \wedge u_i)) = \neg v \wedge u_i.$$

Since $v \wedge \neg v \wedge u_i = 0$, the family $\{a, a_i|_{\neg v \wedge u_i}\}$ is compatible and so it has a unique gluing a' such that $a \subsetneq a'$ and $a' \in \Gamma$, which is impossible by maximality of a. □

Kripke-Joyal semantics allows us to characterize the connected elements of a locale in the following way. This result is not found in [Ca95] and we don't know if it has been noted before.

2.11 Theorem. *$u \in \Omega$ is connected iff for each sheaf of structures $\mathfrak{A}$ on Ω and each formula ϕ, $\mathfrak{A} \Vdash_u (\phi \vee \neg\phi)$ iff $\mathfrak{A} \Vdash_u \phi$ or $\mathfrak{A} \Vdash_u \neg\phi$*

Proof. $\Rightarrow$. Assume u is connected and $\mathfrak{A} \Vdash_u (\phi \vee \neg\phi)$, then $u \leq [\phi^{\mathfrak{A}}] \vee \neg[\phi^{\mathfrak{A}}]$, with $[\phi^{\mathfrak{A}}] \wedge \neg[\phi^{\mathfrak{A}}] = 0$ and so $u \leq [\phi^{\mathfrak{A}}]$, or $u \leq \neg[\phi^{\mathfrak{A}}]$ as we wanted to see.

$\Leftarrow$. Assume that $u = v \vee w$ with $v \wedge w = 0$. Consider the sheaf of sets A such that $A(x) = \{x\}$ for all $x \in \Omega$, it is sometimes denoted by $F(x) = \{*\}$. The restrictions are the only evident. Let R_v be a 1-ary relation symbol, we endow A with an interpretation of R_v in $A(x)$, for all $x \in \Omega$, so that $R_v^{A(x)}(x)$ iff $x \leq v$. The fact that A is a sheaf of structures means simply that if $x = \bigvee_{i\in I} x_i$ and for all $i \in I$, $x_i \leq v$, then $x \leq v$, which is evident. Let's consider now the element $1 \in F(1)$, we have $[R_v^A(1)] = v$, since for $x \in \Omega$, $\mathfrak{A} \Vdash_x R_v(1)$ iff $x \leq v$. In this way $[\neg R_v^A(1)] = \neg v$.

Since we have that $v \wedge w = 0$, we have $w \leq \neg v$, so that $u \leq v \vee \neg v$. By the stated above $\mathfrak{A} \Vdash_u R_v(1) \vee \neg R_v(1)$, and using our hypothesis we have $\mathfrak{A} \Vdash_u R_v(1)$ or $\mathfrak{A} \Vdash_u \neg R_v(1)$. In the first case $u \leq v$, and so $u = v$, in the second case $u \leq \neg v$, and then

$$u = \neg v \wedge u = (\neg v \wedge v) \vee (\neg v \wedge w) = \neg v \wedge w,$$

so $u \leq w$ and then $u = w$. We conclude that u is connected. □

2.12 Corollary. *Let u be in Ω we have:*

1. *u is connected iff for all sheaf of structures on Ω, $\mathfrak{A}$ and ϕ, ψ formulas of τ, if $\mathfrak{A} \Vdash_u (\phi \vee \psi) \wedge \neg(\phi \wedge \psi)$, then $\mathfrak{A} \Vdash_u \phi \wedge \neg\psi$ or $\mathfrak{A} \Vdash_u \psi \wedge \neg\phi$.*

2. *If u is connected* $\mathfrak{A} \Vdash_u (\phi \vee \psi) \to (\neg(\phi \wedge \psi) \leftrightarrow (\neg\phi \vee \neg\psi))$.

Proof. The second assertion is a direct consequence of the first which is proved in a similar way to the above theorem. □

Although we do not offer a characterization of compact locales in terms of the semantics described, this can be achieved in a semantics similar to that of Fourman and Scott, that is, a semantic for Ω-structures in general which don't ask for completeness.In this semantics, the interpretation of each term $t(x_1, \ldots, x_n)$ is given by an Ω-morphism $t^A : A^n \to A$, where the construction of this interpretation follows ours but takes into account that interpretations of constant and function symbols are Ω-morphisms rather than elements and functions respectively. Interpretation of atomic formulas are given by

$$[(t_1(\bar{a}) = t_2(\bar{b}))^A] = \bigvee_{b \in A} [b \sim t_1^A(\bar{a})] \wedge [b \sim t_2^A(\bar{b})].$$

$$[R(t_1(\overline{a_1}), \ldots, t_n(\overline{a_n}))^A] = \bigvee_{\bar{b} \in A^n} [R^A(\bar{b})] \wedge \bigwedge_{i=1}^{n} [b_i \sim t_i(\overline{a_i})].$$

The remaining semantics is defined just as for complete Ω-structures. Using the former semantics, analogous to the one defined in [FS79], it is possible to characterize compact locales in the following way: a locale Ω is compact iff for all Ω-structure $\mathfrak{A}$ and formula ϕ, $[(\exists x \phi(x))^A] = 1$ is equivalent to the existence of $a_1, \ldots, a_n \in A$ such that $1 = \bigvee_{i=1}^{n} [\phi^A(a_i)] \wedge [a_i \approx a_i]$. The implication $\Rightarrow$ is evident and for $\Leftarrow$, if $1 = \bigvee_{i \in I} u_i$ and we want to obtain a finite subcovering of 1 then we consider the Ω-set with universe $\{u_i : i \in I\}$, the Ω-equality given by $[u_i \approx u_j] = u_i \wedge u_j$ and the formula $\exists x (x = x)$.

Following Caicedo's steps we prove the next theorem [Ca95, p. 20].

2.13 Theorem. *Let $\downarrow u$ be an open connected sublocale of Ω, then*

(a) For all $a, b \in A$, $\mathfrak{A} \Vdash_u (a = b \vee \neg a = b)$ iff for all $v \in \Omega$ such that $0 < v \leq u$, ρ_v^u is injective.

(b) $\mathfrak{A} \Vdash_u (\phi \vee \neg\phi)$ for all atomic ϕ iff for all $v \in \Omega$ such that $0 < v \leq u$, ρ_v^u is an injective embedding of structures.

(c) $\mathfrak{A} \Vdash_u (\phi \vee \neg\phi)$ for all formula ϕ iff for all $v \in \Omega$ such that $0 < v \leq u$ ρ_v^u is an elemental equivalence.

Proof. In order to see (a), let a, b be elements in A and suppose $\mathfrak{A} \Vdash_u (a = b \vee \neg a = b)$, then there are $v, w \leq u$ such that $v \vee w = u$ and $w \leq [a \approx b]$, $v \leq \neg[a \approx b]$, so that $w \wedge u = 0$ and by connectedness of u, $w = u$ or $v = u$. Let $0 < u_1 \leq u$ such that $a|_{u_1} = b|_{u_1}$, that is to say, $u_1 \leq [a \approx b]$ then $v \wedge u_1 = 0$ y and so $w = u$. We have $\mathfrak{A} \Vdash_u a = b$. We get then that the restrictions are injective. For the other implication let a, b be in $A(u)$, if there is $0 < v \leq u$ such that $a|_v = b|_v$, then $a = b$, in particular $u \leq [a \approx b]$. If not, then $[a \approx b] = 0$, so $u \leq \neg[a \approx b]$. In any case $\mathfrak{A} \Vdash_u (a = b \vee \neg a = b)$.

(b) follows in the same way that (a), replacing $a = b$ by $R(a_1, \ldots, a_n)$, since the restriction are homomorphisms of structures. We also obtain (c) in the same way that (a) since by the definition of our semantics, if $v \leq u$ and $\mathfrak{A} \Vdash_u \phi$, we have $\Vdash_v \phi$, so, we replace $a = b$ by $\phi(a_1, \ldots, a_n)$ in the proof of (a). □

2.14 Example. [Ca95, p. 20] For the sheaf of germs of holomorphic functions we have that, if two holomorphic functions agree in an open, non empty subset V of an open connected set U, then they are identical in all U. By the former theorem this is equivalent to the fact that the excluded middle holds for atomic formulas in the semantics of this sheaf.

In order to see that our semantics is a semantics for the intuitionistic logic we prove that Kripke Models, a natural semantics for intuitionistic logic, can be seen as sheaves of structures.

2.15 Definition. A *Kripke model* for language τ is a fourfold structure $\mathbb{K} = (\Lambda, \leq, (\mathbb{K}_i), (f_{ij})_{i \leq j})$ which satisfies:

(a) $(\Lambda, \leq)$ is a poset.

(b) For all $i \in \Lambda$, $\mathbb{K}_i$ is a structure with signature τ.

(c) For all $i \leq j \in \Lambda$, $f_{ij} : \mathbb{K}_i \to \mathbb{K}_j$ is a homomorphism of structures.

(d) For all $i \in \Lambda$ f_{ii} is the identity.

(e) For all $i \leq j \leq k$, $f_{jk} \circ f_{ij} = f_{ik}$.

2.16 Definition (Semantics of Kripke models)**.** We define the relation $\mathbb{K} \Vdash_i \phi(\bar{a})$, for all $i \in \Lambda$ and $\bar{a} \in \mathbb{K}_i^n$ by recursion in the complexity of ϕ.

1. If ϕ is an atomic formula $\mathbb{K} \Vdash_i \phi(\bar{a})$ iff $\mathbb{K}_i \models \phi(\bar{a})$.

2. $\mathbb{K} \Vdash_i \phi \vee \psi$ iff $\mathbb{K} \Vdash_i \phi$ or $\mathbb{K} \Vdash_i \psi$.

3. $\mathbb{K} \Vdash_i \phi \wedge \psi$ iff $\mathbb{K} \Vdash_i \phi$ and $\mathbb{K} \Vdash_i \psi$.

4. $\mathbb{K} \Vdash_i \phi(\bar{a}) \rightarrow \psi(\bar{a})$ iff for all $j \geq i$, if we have $\mathbb{K} \Vdash_j \phi(f_{ij}(a_1), \ldots, f_{ij}(a_n))$, then $\mathbb{K} \Vdash_j \psi(f_{ij}(a_1), \ldots, f_{ij}(a_n))$.

5. $\mathbb{K} \Vdash_i \neg\phi(\bar{a})$ iff for all $j \geq i$, $\mathbb{K} \nVdash_j \phi(f_{ij}(a_1), \ldots, f_{ij}(a_n))$.

6. $\mathbb{K} \Vdash_i \exists x \phi(x, \bar{a})$ iff there is $a \in \mathbb{K}_i$ such that $\mathbb{K} \Vdash_i \phi(a, \bar{a})$.

7. $\mathbb{K} \Vdash_i \forall x \phi(x, \bar{a})$ iff for all $j \geq i$ and all $b \in \mathbb{K}_j$ $\mathbb{K} \Vdash_j \phi(b, f_{ij}(a_1), \ldots, f_{ij}(a_n))$.

In [Ca95, p. 22] it is shown how every Kripke model can be seen as a sheaf of structures on a certain topological space and so on its locale of open sets. This result holds for our case then. However, we will use the sheafification technique described in the previous section in order to describe the sheaf of structures associated to each Kripke model.

Given a Kripke model $\mathbb{K}$ we consider the topological space Λ with the topology associated to the order $\Lambda^+ = \{S \subseteq \Lambda \mid \text{ for } i \in S \text{ and } j \leq i, j \in S\}$, with basis $[i) = \{j \in \Lambda \mid j \geq i\}$. For all $i \in I$, let $K[i) = \mathbb{K}_i$ and let's consider the Λ^+-structure with universe $A = \coprod_{i \in I} K[i)$ and such that for all $a \in K[i), b \in K[j)$ and $a_k \in K[i_k)$

- $[a \approx b] = \bigvee\{l \in \Lambda \mid f_{il}(a) = f_{jl}(b)\}$,
- $[R^A(\bar{a})] = \bigvee\{l \in \Lambda \mid R^{K[l)}(f_{i_1 l}(a_1), \ldots, f_{i_n l}(a_n))\}$,
- $[a \sim c^{\mathfrak{A}}(*)] = \bigvee_{l \in \Lambda}[a \approx c^{K[l)}]$,
- $[a \sim g^{\mathfrak{A}}(\bar{a})] = \bigvee\{l \in \Lambda \mid f_{il}(a) = g^{K[l)}(f_{i_1 l}(a_1), \ldots, f_{i_n l}(a_n))\}$,

where c, R and g are symbols in τ. Consider now the complete Ω-structure given by $\sigma(A)$, and the sheaf of structures $\Sigma \circ \sigma(A) = \mathbb{K}^*$. Let's see that σ induces an isomorphism of structures between $\mathbb{K}^*([i))$ and $K([i))$. The fact that it preserves relations and functions follows quite easily using calculations already employed in the definition of the functors Γ and Σ. We prove bijectivity.

For injectivity, let a, b be elements of $K[i)$ and assume $\sigma_a = \sigma_b$, then we have $[a \approx b] = [a \approx a] = [b \approx a] = [i)$. Now, we have

$$[i) = \bigcup\{[j) \mid j \geq i \text{ and } f_{ij}(a) = f_{ij}(b)\},$$

but, if $i \in [j)$, with j in the former union, then $i = j$ and we have that $a = f_{ii}(a) = f_{ii}(b) = b$. For surjectivity, if ρ is a singleton such that $[i) = \bigcup_{a \in A} \rho(a)$, and $i \in \rho(a)$, then $[i) \subseteq \rho(a) \subseteq [i)$, and since $\rho(a) \leq [a \approx a]$, for all $a \in K[k]$, then $k \leq i$ and we can consider the restriction $b = f_{ki}(a)$. Then we have $[a \approx b] = [i)$ and we infer

$$\rho(a) \cap [a \approx b] = [i) \subseteq \rho(b) \subseteq [b \approx b] = [i).$$

We conclude that $b \in K[i]$ and $\rho(b) = [i)$. Let's see that $\sigma_b = \rho$, let c be in A, we have to show $\rho(c) = [b \approx c]$. Since $\sigma(c) \subseteq [i) = \rho(b)$, we have $\rho(c) = \rho(c) \cap \rho(b) \leq [b \approx c]$. In a similar way, using that $[b \approx c] \leq [b \approx b] = [i)$, we have $[b \approx c] = [b \approx c] \wedge \rho(b) \leq \rho(c)$. The fact that the inverse of σ preserves relations follows easily. We have that σ induces an isomorphism of structures between $\mathbb{K}_i$ and $\mathbb{K}^*([i))$. This proves that $\mathbb{K}^*$ coincides with the sheaf of structures defined in [Ca95, p. 22]. We can see that for all $i \leq j \in \Lambda$

$$\begin{array}{ccc} \mathbb{K}_i & \xrightarrow{f_{ij}} & \mathbb{K}_j \\ \sigma\downarrow & & \downarrow\sigma \\ \mathbb{K}^*[i) & \xrightarrow[\rho^{[i)}_{[j)}]{} & \mathbb{K}^*[j) \end{array} \quad .$$

Semantics of a Kripke model can be captured in its associated sheaf of structures in the following way, as in [Ca95, p. 23].

2.17 Lemma. *Let $\mathbb{K}$ be a Kripke model and $\mathbb{K}^*$ its associated sheaf of structures. The following are equivalent*

1. $\mathbb{K} \Vdash_i \phi(a_1, \ldots, a_n)$ *in the semantics for Kripke models.*

2. $\mathbb{K}^* \Vdash_{[i)} \phi(\sigma_{a_1}, \ldots, \sigma_{a_n})$ *in the Kripke-Joyal semantics.*

Proof. We reason by induction in the complexity of ϕ. If ϕ is an atomic formula, since σ induces an isomorphism of structures between $\mathbb{K}_i$ and $\mathbb{K}^*([i))$, the Kripke semantics coincides in the atomic case with the classic semantics of $\mathbb{K}_i$ and Kripke-Joyal semantics agree with the classic semantics of $\mathbb{K}^*([i))$. The inductive step is trivial for the connective $\wedge$. For connectives $\neg, \Rightarrow$ y $\forall$ we note that for all $i, j \in \Lambda$, $i \leq j$ is equivalent to $[j) \subseteq [i)$. For $\vee$ and $\exists$ we only prove $(2) \rightarrow (1)$. For the connective $\vee$, if there are opens u, v such that $u \cup v = [i)$ and we have $\mathbb{K}^* \Vdash_u \phi$ and $\mathbb{K}^* \Vdash_u \psi$, then, if $i \in u$, $\mathbb{K} \Vdash_i \phi$, and similarly for $i \in v$. For the connective $\exists$ if there is an open covering $\{u_j\}_{j \in J}$ of $[i)$ and elements $a_j \in u_j$ such that $\mathbb{K}^* \Vdash_{u_j} \phi(a_j, \sigma_{b_1}, \ldots, \sigma_{b_n)})$, then, for all $i \in u_j$, $\mathbb{K}^* \Vdash_{[i)} \phi(a_j, \sigma_{b_1}, \ldots, \sigma_{b_n)})$, and by surjectivity of σ and inductive hypothesis, there is $a \in \mathbb{K}_i$ such that $\Vdash_i \phi(a, b_1, \ldots, b_n)$. $\square$

2.18 Theorem. $\vdash_{H^*} \phi$ *iff for all locale* Ω *and all sheaf of structures on* Ω $\mathfrak{A}$, $\mathfrak{A} \models_1 \phi$.

Proof. If ϕ has free variables, we consider the universal closure of ϕ instead of ϕ itself. Let ϕ be a formula such that for all locale Ω and all sheaf of structures on Ω $\mathfrak{A}$, $\mathfrak{A} \models_1 \phi$. By the previous lemma ϕ is satisfied in all Kripke models and since these constitute a semantic for the intuitionistic logic H^* [vD02, p. 33], we have $H^* \vdash \phi$. For the other implication, that is to say, the correctness of the logic of sheaves of structure we note that axioms H1-H10 are satisfied in all sheaf of structures on a locale Ω since Ω is a Heyting algebra, which also implies the satisfaction of the Modus Ponens rule of deduction. Using simple calculations we realize the satisfaction of the axioms in the propositional calculus and the correctness of the derivative rules in our semantics. □

In what follows we prepare the proof of the generic model theorem [Ca95, p. 28].

2.19 Definition. A *filter* $\mathcal{F}$ in a locale Ω is $\mathcal{F} \subseteq \Omega$ such that

- $1 \in \mathcal{F}$.
- For all $u, v \in \mathcal{F}$, $u \wedge v \in \mathcal{F}$.
- If $v \in \mathcal{F}$ and $v \leq u$, then $u \in \mathcal{F}$.

We say that the filter $\mathcal{F}$ is trivial when $0 \in \mathcal{F}$. A non trivial filter is maximal when it is maximal for the order $\subseteq$ among non trivial filters. Just as in the case of filters of open sets in topological spaces we have:

2.20 Lemma. *1. Let* $\mathcal{F}$ *be a maximal filter, then* $w \in \mathcal{F}$ *iff for all* $u \in \mathcal{F}$, $w \wedge u > 0$.

2. $\mathcal{F}$ *is a maximal filter iff for all* $u \in \Omega$, $u \in \mathcal{F}$ *or* $\neg u \in \mathcal{F}$.

3. Let $\mathcal{F}$ *be a maximal filter and* $w \in \Omega$ *such that there is* $u \in \mathcal{F}$ *which satisfies* $\neg w \wedge u = 0$, *then* $w \in \mathcal{F}$.

Proof. For 2. we note that if $\mathcal{F}$ is maximal and $u \notin \mathcal{F}$, then, by 1. there is $w \in \mathcal{F}$ such that $u \wedge w = 0$, that is to say, $w \leq \neg u$, so $u \in \mathcal{F}$. Now, if we assume that for all $u \in \Omega$, $u \in \mathcal{F}$ or $\neg u \in \mathcal{F}$ and $\mathcal{F} \subseteq \mathcal{G}$ with G a non trivial filter, if $v \in \mathcal{G}$ and $\neg v \in \mathcal{F}$, then $v \wedge \neg v = 0 \in \mathcal{G}$, which is impossible. We conclude $v \in \mathcal{F}$.

3. follows directly from 1. and 2. □

The following definition is used to state the theorem.

2.21 Definition (Generic filter for a sheaf of structures)**.** Let $\mathfrak{A}$ be a sheaf of structures on Ω. A non trivial filter $\mathcal{F} \subseteq \Omega$ is generic for $\mathfrak{A}$ when for all $u \in \mathcal{F}$, a_i such that $[a_i \approx a_i] \geq u$ and formulas ϕ, ψ we have that:

1. There is $w \in \mathcal{F}$ such that $\mathfrak{A} \Vdash_w \phi(\bar{a})$ or there is $w \in \mathcal{F}$ such that $\mathfrak{A} \Vdash_w \neg\phi(\bar{a})$.
2. If $\mathfrak{A} \Vdash_u \exists x\psi(x, \bar{a})$, then there is $w \in \mathcal{F}$ and $b \in A$ satisfying $w \leq [b \approx b]$ such that $\mathfrak{A} \Vdash_w \psi(b, \bar{a})$.

Maximal filters are our natural examples of generic filters so we will be using axiom of choice.

2.22 Theorem. *Let $\mathcal{F} \subseteq \Omega$ be a non trivial filter. $\mathcal{F}$ is a maximal filter iff $\mathcal{F}$ is generic for all sheaves of structures on Ω.*

Proof. $\Rightarrow$. Let $\mathcal{F}$ be a maximal filter, $u \in \mathcal{F}$, a_i with $[a_i \approx a_i] \geq u$ and formulas ϕ, ψ. By maximality of $\mathcal{F}$ and the previous lemma, $[\phi(\bar{a})] \in \mathcal{F}$ or $\neg[\phi(\bar{a})] \in \mathcal{F}$. In the first case $\mathfrak{A} \Vdash_{[\phi(\bar{a})]} \phi(\bar{a})$ and in the other case $\mathfrak{A} \Vdash_{\neg[\phi(\bar{a})]} \neg\phi(\bar{a})$. If $\Vdash_u \exists x\psi(x, \bar{a})$, by the maximum principle there is $w \leq v$ such that $\neg w \wedge v = 0$ and $\mathfrak{A} \Vdash_w \psi(b, \bar{a})$. Thus, we can infer $w \in \mathcal{F}$ and $[b \approx b] \geq w$.

$\Leftarrow$. Let's consider the sheaf of sets A such that $A(w) = \{w\}$ for $w \in \Omega$, sometimes denoted $F(w) = \{*\}$, with the unique restrictions possible. Let u be an element of Ω, we consider the 1-ary relation symbol R_u and endow $A(w)$, with the same structure as in theorem **2.11** so that $R_u^{A(w)}(w)$ iff $w \leq u$. Now consider $1 \in F(1)$. Since $\mathcal{F}$ is generic for this sheaf of structures $\mathfrak{A}$, there is $w \in \mathcal{F}$ such that $\mathfrak{A} \Vdash_w R_u(w)$ or $\mathfrak{A} \Vdash_w \neg R_u(w)$. Note that $(1|_w = w)$. In the first case we have $R_u^{A(w)}(w)$, that is $w \leq u$ so $u \in \mathcal{F}$. In the second case we have that for all v satisfying $0 < v \leq w$, we do not have $v \leq u$. This implies that $w \wedge u = 0$, that is $w \leq \neg u$, so that $\neg u \in \mathcal{F}$. By the previous lemma we conclude that $\mathcal{F}$ is maximal. $\square$

We need to define the notion of filtered limit in the category of first order structures of signature τ.

2.23 Definition (Filtered limit of structures)**.** Let Ω be a locale, $\mathcal{F} \subseteq \Omega$ a nontrivial filter. Let $\mathfrak{A} : \Omega \to St_\tau$ be a presheaf of structures. The classic limit $\mathfrak{A}[\mathcal{F}] = \varinjlim_{u \in \mathcal{F}} A(u)$ is a structure with signature τ whose universe is the set of equivalence classes of the relation $\sim_\mathcal{F}$ defined in the disjoint union $\coprod_{u \in \mathcal{F}} A(u)$ in the following way. For all $a \in A(u)$and all $b \in A(v)$,

$$a \sim b \text{ iff there is } w \in \mathcal{F} \text{ such that } w \leq u \wedge v \text{ and } a|_w = b|_w.$$

We give $\mathfrak{A}$ structure in such a way that if $a_i \in A(u_i)$, $[a_i]$ is the corresponding equivalence class and c, f, R are symbols of constant, function and relation:

- $c^{\mathfrak{A}[\mathcal{F}]} = c^{A(1)}$,
- $R^{\mathfrak{A}[\mathcal{F}]}([a_1], \ldots, [a_n])$ iff there is $u \in \mathcal{F}$, with $u \leq \bigwedge_{i=1}^{n} u_i$ such that $R^{A(u)}(a_1|_u, \ldots, a_n|_u)$.
- $f^{\mathfrak{A}[\mathcal{F}]}([a_1], \ldots, [a_n]) = f^{A(u)}(a_1|_u, \ldots, a_n|_u)$, where $u = \bigwedge_{i=1}^{n} u_i$.

When $\mathcal{F}$ is a generic filter for a sheaf of structures $\mathfrak{A}$, we say that $\mathfrak{A}[\mathcal{F}]$ is a generic model. We need to define the Gdel translation [Ca95, p. 21]. This translation was introduced by Gdel for proving that classic arithmetic is interpretable in intuitionistic arithmetic, showing the equiconsistency of both.

2.24 Definition (Gdel's translation)**.** For each formula ϕ we define its Gdel translation ϕ^G by recursion in the complexity of ϕ in the following way:

- For an atomic ϕ, $\phi^G := \neg\neg\phi$.
- $(\phi \vee \psi)^G := \neg(\neg\phi^G \wedge \neg\psi^G)$.
- $(\phi \wedge \psi)^G := \phi^G \wedge \psi^G$.
- $(\phi \rightarrow \psi)^G := \neg(\phi^G \wedge \neg\psi^G)$.
- $\neg(\phi)^G := \neg\phi^G$.
- $(\forall x\phi)^G := \forall x\phi^G$.
- $(\exists x\phi)^G := \neg\forall x(\neg\phi^G)$.

The following theorem is considered by Caicedo as the Fundamental Theorem of Model Theory since it has as corollaries Łoś Ultraproducts Theorem, the Ommiting Types Theorem and the Completeness of first order logic [Ca95, p. 28]. Caicedo also shows that forcing is interpretable in terms of sheaves of structures. With this theorem we end this text.

2.25 Theorem (Generic model theorem)**.** *Let $\mathcal{F} \subseteq \Omega$ be a generic filter for $\mathfrak{A}$, then the following are equivalent:*

1. $\mathfrak{A}[\mathcal{F}] \models \phi([a_1], \ldots, [a_n])$.
2. $[(\phi^G)^{\mathfrak{A}}(a_1, \ldots, a_n)] \in \mathcal{F}$.
3. *There is $u \in \mathcal{F}$ such that $\mathfrak{A} \Vdash_u \phi^G(a_1, \ldots, a_n)$.*

Proof. By induction in the complexity of ϕ. For the atomic case we have

$$[a_1] = [a_2] \text{ iff exists } w \in \mathcal{F} \text{ such that } a_1|_w = a_2|_w$$
$$\text{iff exists } w \in \mathcal{F} \text{ such that } \mathfrak{A} \Vdash_w (a_1 = a_2)$$
$$\text{iff exists } w \in \mathcal{F} \text{ such that } \mathfrak{A} \Vdash_w \neg\neg(a_1 = a_2),$$

where in the last equivalence we have implication $\Leftarrow$ since, using that $\mathcal{F}$ is generic, we have the existence of an element $u \in \mathcal{F}$ such that $\mathfrak{A} \Vdash_u a_1 = a_2$ or $\mathfrak{A} \Vdash_u \neg(a_1 = a_2)$. We can discard the last option since, in such case $w \wedge u = 0$. For the case where ϕ is $R([a_1], \dots, [a_n])$ we reason analogously. For the logical connective $\wedge$ we have:

$$\mathfrak{A}[\mathcal{F}] \models \phi \wedge \psi \text{ iff } \mathfrak{A}[\mathcal{F}] \models \phi \text{ and } \mathfrak{A}[\mathcal{F}] \models \psi$$
$$\text{iff there are } u, v \in \mathcal{F} \text{ such that } \mathfrak{A} \Vdash_u \phi^G \text{ and } \mathfrak{A} \Vdash_v \psi^G$$
$$\text{iff there is } w \in \mathcal{F} \text{ such that } \mathfrak{A}_w \Vdash \phi^G \wedge \psi^G.$$

For the logical connective $\vee$ we have that

$$\mathfrak{A}[\mathcal{F}] \models \phi \vee \psi \text{ iff } \mathfrak{A}[\mathcal{F}] \models \phi \text{ or } \mathfrak{A}[\mathcal{F}] \models \psi$$
$$\text{iff there is } v \in \mathcal{F} \text{ such that } \mathfrak{A} \Vdash_v \phi^G \text{ or there is } v \in \mathcal{F} \text{ such that } \mathfrak{A} \Vdash_v \psi^G$$
$$\text{iff there is } v \in \mathcal{F} \text{ such that } \mathfrak{A} \Vdash_v \phi^G \vee \psi^G$$
$$\text{iff there is } v \in \mathcal{F} \text{ such that } \mathfrak{A} \Vdash_v \neg\neg(\phi^G \vee \psi^G)$$
$$\text{iff there is } v \in \mathcal{F} \text{ such that } \mathfrak{A} \Vdash_v \neg(\neg\phi^G \wedge \neg\psi^G),$$

where the last equivalence holds since Kirpke-Joyal semantics satisfies intuitionistic logic and the third since, if $w \in \mathcal{F}$ satisfies $\mathfrak{A} \Vdash_w \phi^G \vee \psi^G$, given that there is $u \in \mathcal{F}$ such that $\mathfrak{A} \Vdash_u \phi^G$ or $\mathfrak{A} \Vdash_u \neg\phi^G$, in the first case there is nothing to prove and in the second one $\mathfrak{A} \Vdash_{w \wedge u} \psi^G$. For the logical connective $\neg$ the situation is simpler:

$$\mathfrak{A}[\mathcal{F}] \models \neg\phi \text{ iff it does not hold that } \mathfrak{A}[\mathcal{F}] \models \phi$$
$$\text{iff for all } w \in \mathcal{F}, \mathfrak{A} \nVdash_w \phi^G$$
$$\text{iff there is } w \in \mathcal{F} \text{ such that } \mathfrak{A} \Vdash_w \neg\phi^G,$$

where the last equivalence holds because $\mathcal{F}$ is closed under $\wedge$ and there is $w \in \mathcal{F}$ such that $\mathfrak{A} \Vdash_w \phi^G$ or $\mathfrak{A} \Vdash_w \phi^G$. For the connective $\rightarrow$ we have

$$\mathfrak{A}[\mathcal{F}] \models \phi \rightarrow \psi \text{ iff } \mathfrak{A}[\mathcal{F}] \models \neg\phi \vee \psi$$
$$\text{iff there is } w \in \mathcal{F} \text{ such that } \mathfrak{A} \Vdash_w \neg\phi^G \vee \psi^G$$
$$\text{iff there is } w \in \mathcal{F} \text{ such that } \mathfrak{A} \Vdash_w \neg(\phi^G \wedge \neg\psi^G),$$

where the first equivalence holds by using classical logic, the second one by the proved before and inductive hypothesis and, for the third, implication $\Rightarrow$ follows from closure of $\mathcal{F}$ under $\wedge$ and the existence of an element $w \in \mathcal{F}$ such that $\mathfrak{A} \Vdash_w \neg(\phi^G \wedge \neg\psi^G)$ or $\mathfrak{A} \Vdash_w (\phi^G \wedge \neg\psi^G)$, being this last option discarded. For $\Leftarrow$ we discard the existence of a w in $\mathcal{F}$ such that $\mathfrak{A} \Vdash_w \neg(\neg\phi^G \vee \psi^G)$, since by laws of the intuitionistic calculus we would have $\mathfrak{A} \Vdash_w \neg\neg\phi^G \wedge \neg\psi^G$, and using the previously proved there would exist some $u \in \mathcal{F}$ such that $\mathfrak{A} \Vdash_u \phi^G \wedge \neg\psi^G$, and that would contradict our hypothesis or the closure of $\mathcal{F}$ under $\wedge$. For the logical connective $\exists$ we have

$$\begin{aligned}
\mathfrak{A}[\mathcal{F}] \models \exists x\phi(x) \quad &\text{iff} \quad \text{there are } w \in \mathcal{F}, a \in A(w) \text{ such that } \mathfrak{A}[\mathcal{F}] \models \phi([a]) \\
&\text{iff} \quad \text{there are } w \in \mathcal{F}, a \in A(w), u \in \mathcal{F} \text{ tal que } \mathfrak{A} \Vdash_u \phi^G(a) \\
&\text{iff} \quad \text{there are } w \in \mathcal{F}, a \in A(w) \text{ such that } \mathfrak{A} \Vdash_w \phi^G(a) \\
&\text{iff} \quad \text{there is } w \in \mathcal{F} \text{ such that } \mathfrak{A} \Vdash_w \exists x\phi^G(x) \\
&\text{iff} \quad \text{there is } w \in \mathcal{F} \text{ such that } \mathfrak{A} \Vdash_w \neg\neg\exists x\phi^G(x) \\
&\text{iff} \quad \text{there is } w \in \mathcal{F} \text{ such that } \mathfrak{A} \Vdash_w \neg\forall x\neg\phi^G(x).
\end{aligned}$$

For the logical connective $\forall$ we have

$$\begin{aligned}
\mathfrak{A}[\mathcal{F}] \models \forall x\phi(x) \quad &\text{iff} \quad \text{for all } w \in \mathcal{F} \text{ and } a \in A(w), \mathfrak{A}[\mathcal{F}] \models \phi([a]) \\
&\text{iff} \quad \text{for all } w \in \mathcal{F} \text{ and } a \in A(w), \text{ there is } u \in \mathcal{F} \text{ such that } \mathfrak{A} \Vdash_u \phi^G(a) \\
&\text{iff} \quad \text{there is } w \in \mathcal{F} \text{ such that for all } v \leq w \text{ and } a \in A(v) \mathfrak{A} \Vdash_v \phi^G(a) \\
&\text{iff} \quad \text{there is } w \in \mathcal{F} \text{ such that } \mathfrak{A} \Vdash_w \forall x\phi^G(x),
\end{aligned}$$

where all the equivalence are evident with the exception of the third one. For $\Leftarrow$ let $w \in \mathcal{F}$ be such that $\mathfrak{A} \Vdash_w \forall x\phi^G(x)$, $u \in \mathcal{F}$ and $a \in A(u)$, then $\mathfrak{A} \Vdash_{w\wedge v} \phi^G(a)$. For $\Rightarrow$ lets assume that $\mathfrak{A}[\mathcal{F}] \models \neg\exists x\neg\phi(x)$, by the previously proved and by inductive hypothesis there is $w \in \mathcal{F}$ such that $\mathfrak{A} \Vdash_w \neg\neg\forall x\neg\neg\phi^G(x)$. Lets assume there is $v \in \mathcal{F}$ such that $\mathfrak{A} \Vdash_v \neg\forall x\phi^G(x)$. By closure of $\mathcal{F}$ under $\wedge$ se can also assume $v = w$, and so $\mathfrak{A} \Vdash_w \neg\neg\forall x\neg\neg\phi^G(x) \wedge \neg\forall x\phi^G(x)$ using the implication already proved we have that $\mathfrak{A}[\mathcal{F}] \models (\forall x\neg\neg\phi) \wedge (\neg\forall x\phi(x))$ which is contradictory. Hence, there is $v \in \mathcal{F}$ such that $\mathfrak{A} \Vdash_v \forall x\phi^G(x)$, as we wanted. This concludes the proof of the theorem. $\square$

CONCLUSIONS

- The category of sheaves of structures on a locale is a reflective subcategory of the category of presheaves of structures on the locale. We gave a proof of this based on the concept of Ω-set and Ω-structure as these are found in Fourman and Scott work [FS79].
- Under some restrictions, the category of Ω-structures studied by Fourman and Scott turns out to be equivalent to the one of sheaves of structures on Ω. Its full subcategory whose objects are complete Ω-structures turns out to be isomorphic to the category of sheaves of structure on Ω, which is a natural generalization of the category studied by Caicedo in [Ca95].
- The results obtained by Caicedo for the logic of sheaves of structures on topological spaces generalize in a natural way to the scope of sheaves on locales, including the Generic Model Theorem.
- We can give a characterization of connected locales, and connected topological spaces in consequence, in terms of the Kripke-Joyal semantics for sheaves of structures.

INDEX

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